National
Qualifications

X847/76/12

Mathematics
Paper 2

Duration — 1 hour 30 minutes

2021 PAPER 2 – WORKED SOLUTIONS

HWallace

Total marks — 65

SECTION 1 — 52 marks

Attempt ALL questions.

SECTION 2 — 13 marks

Attempt EITHER Part A OR Part B.

1. Determine the equation of the tangent to the curve $y = 2x^3 - 8x^2 + 14$ at the point where $x = 3$. **4**

$y = 2x^3 - 8x^2 + 14$

$\frac{dy}{dx} = 6x^2 - 16x$ ✓

Gradient @ $x = 3$

$m = 6(3)^2 - 16(3)$

$m = 6(9) - 48$

$m = 54 - 48$

$m = 6$ ✓

Point of Tangency:

$y = 2(3)^3 - 8(3)^2 + 14$

$y = 2(27) - 8(9) + 14$

$y = 54 - 72 + 14$

$y = -4$ ✓ P.O.T. $(3, -4)$

Equation of tangent

$y - b = m(x - a)$

$y + 4 = 6(x - 3)$

$y + 4 = 6x - 18$

$y = 6x - 22$ ✓

2. Find $\int \frac{6}{(x+5)^{\frac{3}{2}}} dx, x > -5.$ **3**

prepare!

$= \int 6(x+5)^{-3/2} \, dx$ ✓

$= \frac{6(x+5)^{-1/2}}{-1/2 \times 1} + c$ ✓

$= -12(x+5)^{-1/2} + c$ ✓

$= -\frac{12}{\sqrt{x+5}} + c$

3. Given $h(t)=\sin\left(2t+\frac{\pi}{6}\right)$, determine the rate of change of h when $t=10$. 3

$$h(t) = \sin\left(2t + \frac{\pi}{6}\right)$$

$$h'(t) = 2\cos\left(2t + \frac{\pi}{6}\right)$$ ✓✓

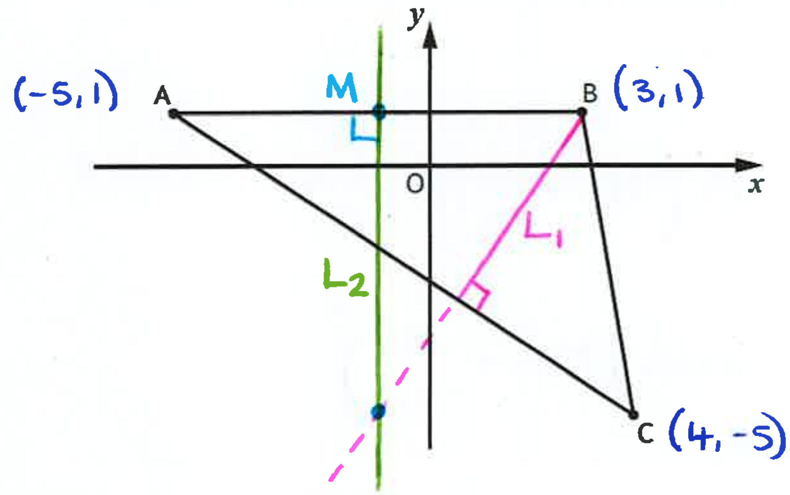
$$h'(10) = 2\cos\left(2(10) + \frac{\pi}{6}\right)$$

$$= 2\cos\left(20 + \frac{\pi}{6}\right)$$ radians!

$$= -0.206126\ldots$$ ✓

Rate of change is -0.206.

4. Triangle ABC has vertices A (–5, 1), B (3, 1) and C (4, –5).



(a) The line L_1 is the altitude through B.

Find the equation of L_1. **3**

(b) The line L_2 is the perpendicular bisector of AB.

Find the equation of L_2. **3**

(c) Determine the coordinates of the point of intersection of L_1 and L_2. **1**

a) Altitude L_1

Gradient of AC

$$m_{AC} = \frac{-5-1}{4-(-5)} = \frac{-6}{9} = -\frac{2}{3}$$

Gradient of L_1

$$m_{L_1} = \frac{3}{2}$$

since $m_1 \times m_2 = -1$

Equation of L_1

$$y - b = m(x-a)$$
$$y - 1 = \frac{3}{2}(x-3)$$
$$2y - 2 = 3x - 9$$
$$2y = 3x - 7$$

4) b) Perpendicular Bisector L_2

$$m_{AB} = \frac{1-1}{3-(-5)} = \frac{0}{8} = 0$$

AB is horizontal $\therefore$ L_2 is vertical!

Midpoint of AB

$$M\left(\frac{-5+3}{2}, \frac{1+1}{2}\right)$$

$$M(-1, 1)$$

Equation of L_2

$$x = -1$$

c) Point of Intersection

L_1 : $2y = 3x - 7$

L_2 : $x = -1$

$$2y = 3(-1) - 7$$
$$2y = -3 - 7$$
$$2y = -10$$
$$y = -5$$

Point $(-1, -5)$

$5\sin t + 3\cos t$

5. (a) Express $3\cos t° + 5\sin t°$ in the form $k\sin(t+a)°$, $k>0$, $0<a<360$. **4**

(b) A function, f, is defined by $f(t) = 3\cos t° + 5\sin t°$, $0 \le t < 360$.

(i) State the minimum value of $f(t)$. **1**

(ii) Determine the value of t where this minimum occurs. **1**

a) $k\sin(t+a)° = k\sin t\cos a + k\cos t\sin a$ ✓

$= k\cos a . \sin t + k\sin a . \cos t$

$\rightarrow 5 . \sin t + 3 . \cos t$

$k\sin a = 3$

$k\cos a = 5$ ✓

$k = \sqrt{3^2+5^2}$

$k = \sqrt{9+25}$

$k = \sqrt{34}$ ✓

$\tan a = \frac{k\sin a}{k\cos a}$

$\tan a = \frac{3}{5}$ S A T C

$ra = \tan^{-1}\left(\frac{3}{5}\right) = 30.96...$

$a = 30.96°$

↳ $3\cos t° + 5\sin t° = \sqrt{34}\sin(t+30.96)°$ ✓

b) $f(t) = 3\cos t° + 5\sin t°$

$f(t) = \sqrt{34}\sin(t+30.96)°$

Amplitude $\sqrt{34}$

shift left by $30.96°$

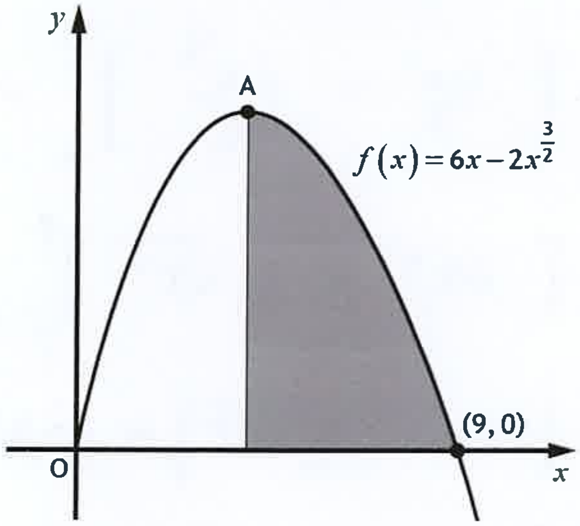
i) Minimum value $= -\sqrt{34}$ ✓

ii) $x = 270 - 30.96$

$x = 239.04°$ ✓

6. The graph of the function $f(x) = 6x - 2x^{\frac{3}{2}}$, $x \geq 0$ is shown.

The point A is a stationary point of $f(x)$.



(a) Determine the x-coordinate of the stationary point A. **3**

(b) Hence calculate the shaded area. **4**

a) Stationary Points occur at $f'(x) = 0$

$$f(x) = 6x - 2x^{3/2}$$

$$f'(x) = 6 - 3x^{1/2} \checkmark$$

$$0 = 6 - 3\sqrt{x} \checkmark$$

$$3\sqrt{x} = 6$$

$$\sqrt{x} = 2$$

$$x = 4 \checkmark$$

6) b) $\text{Area} = \int_4^9 6x - 2x^{3/2} \, . dx$ ✓

$$= \left[\frac{6x^2}{2} - \frac{2x^{5/2}}{5/2} \right]_4^9 \checkmark$$

$$= \left[3x^2 - \frac{4}{5}\sqrt{x^5} \right]_4^9$$

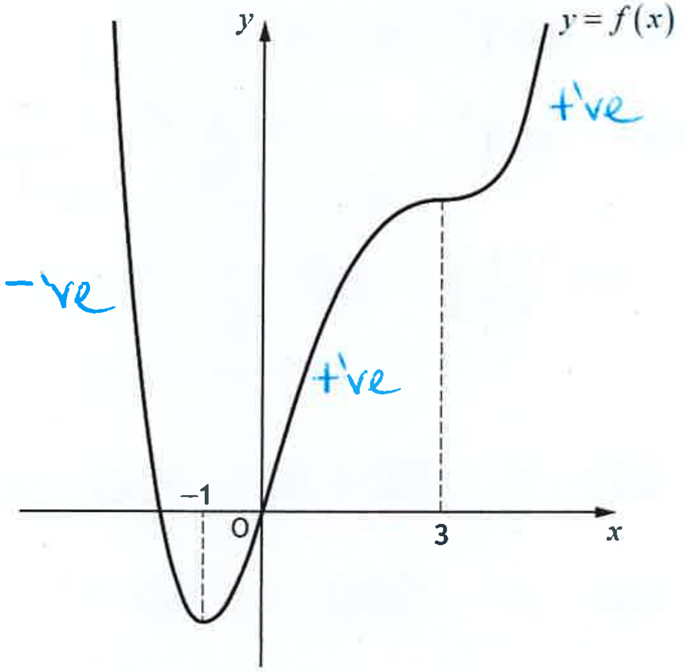
$$= \left(3(9)^2 - \frac{4}{5}\sqrt{9^5}\right) - \left(3(4)^2 - \frac{4}{5}\sqrt{4^5}\right) \checkmark$$

$$= \left(3(81) - \frac{4}{5}(243)\right) - \left(3(16) - \frac{4}{5}(32)\right)$$

$$= \left(243 - \frac{972}{5}\right) - \left(48 - \frac{128}{5}\right)$$

$\text{Area} = 26\frac{1}{5} \text{ units}^2$ ✓

7. The diagram shows the graph of $y = f(x)$, which has stationary points at $x = -1$ and $x = 3$.

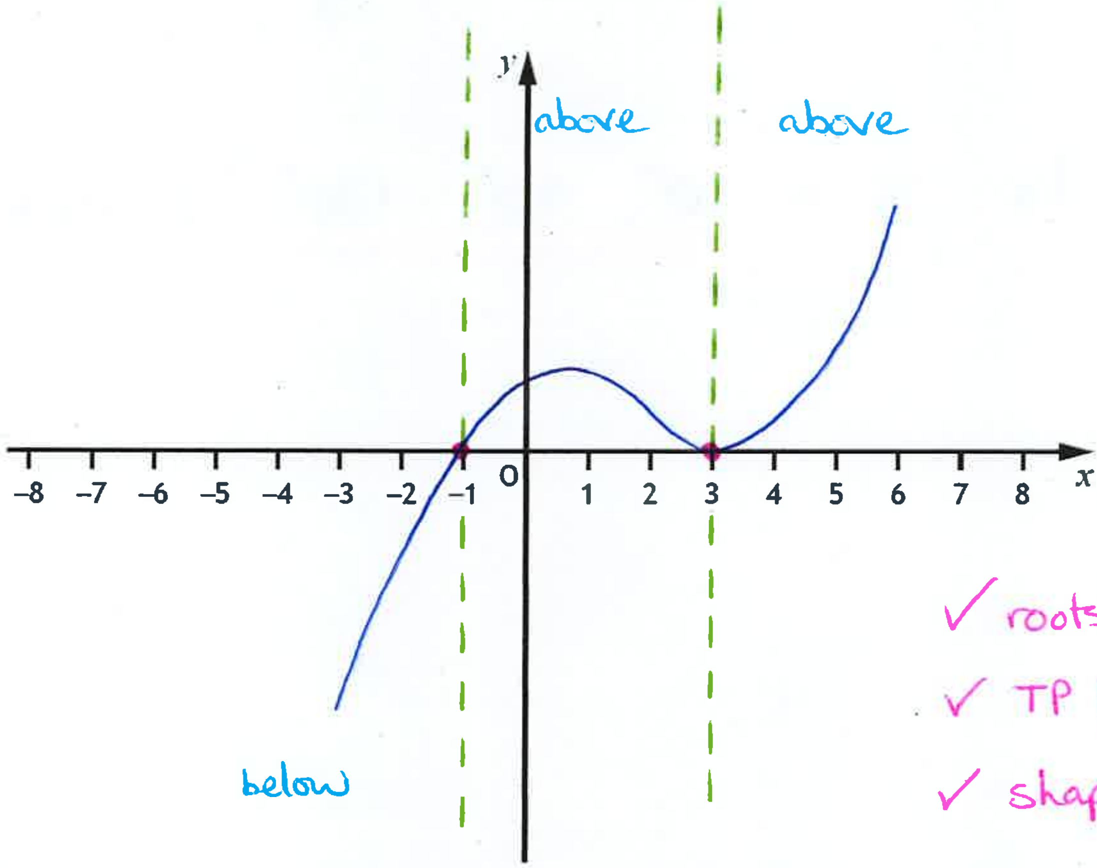


Positive gradient
$\therefore f'(x) > 0$

Negative gradient
$\therefore f'(x) < 0$

On the diagram in your answer booklet, sketch a possible graph of $y = f'(x)$. **3**

Stationary Points on $y = f(x)$ lead to roots on $y = f'(x)$.

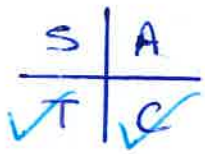


✓ roots
✓ TP (3,0)
✓ shape

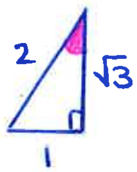
8. Solve the equation $2\sin(3x-60)^\circ+1=0$, $0 \le x < 180$. **4**

$$2\sin(3x-60)^\circ = -1$$

$$\sin(3x-60)^\circ = -\frac{1}{2}$$



$$ra = \sin^{-1}\left(\tfrac{1}{2}\right) = 30^\circ$$



↳ $3x - 60 = 180+30,\ 360-30$

$3x - 60 = 210,\ 330$

$3x = 270,\ 390$

$x = 90,\ 130$

-120

3 waves : Period $= \frac{360}{3} = 120^\circ$.

↳ $x = 10^\circ,\ 90^\circ,\ 130^\circ$ for $0 \le x < 180$

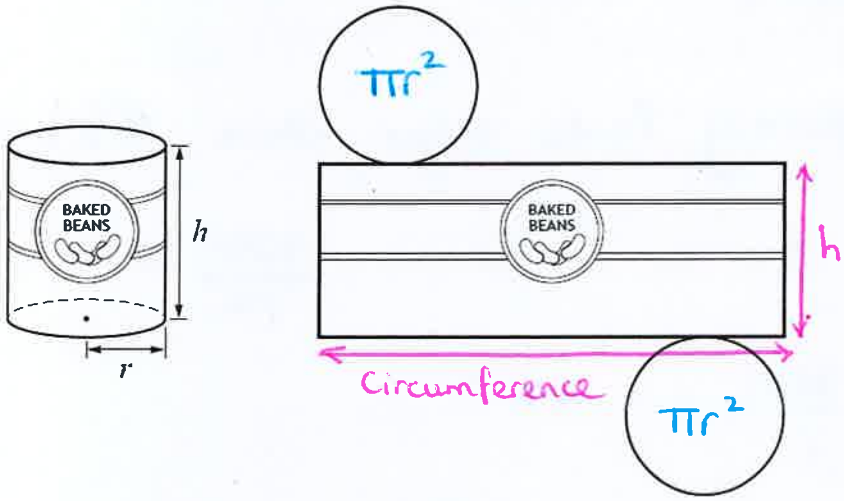
9. A cylindrical tin of baked beans has a volume of 450 cm^3.

The radius of the tin is r cm and its height is h cm.

A net of the tin is shown in the diagram.

$C = \pi d$

$C = 2\pi r$



(a) Show that the surface area of the tin, A square centimetres, is given by

$$A(r) = 2\pi r^2 + \frac{900}{r}.$$ **3**

(b) Determine the radius that will minimise the surface area. **6**

a) Total Surface Area = 2 circles + rectangle

$= 2\pi r^2 + 2\pi r h$ ✓

Volume $= \pi r^2 h$

$450 = \pi r^2 h$

$h = \dfrac{450}{\pi r^2}$ ✓

$\hookrightarrow \quad A(r) = 2\pi r^2 + 2\pi r\left(\dfrac{450}{\pi r^2}\right)$ ✓

$A(r) = 2\pi r^2 + \dfrac{900}{r}$ as required.

9) b) $A(r) = 2\pi r^2 + 900r^{-1}$ ✓

$A'(r) = 4\pi r - 900r^{-2}$ ✓

Stationary Points occur when $A'(r) = 0$

$0 = 4\pi r - \frac{900}{r^2}$ ✓

$\frac{900}{r^2} = 4\pi r$

$900 = 4\pi r^3$

$r^3 = \frac{900}{4\pi}$

$r = \sqrt[3]{\frac{225}{\pi}}$ ✓

$r \approx 4.15$ cm

Nature Table

r	$\xrightarrow{4}$	$\sqrt[3]{\frac{225}{\pi}}$	$\xrightarrow{5}$
$A'(r)$	$-$	0	$+$
slope	\	_	/

✓

$A'(4) = 4\pi(4) - \frac{900}{16} = -$'ve

$A'(5) = 4\pi(5) - \frac{900}{25} = +$'ve

Surface Area will minimise when $r = \sqrt[3]{\frac{225}{\pi}}$ cm ✓

10. (a) Show that $2\tan x\cos^2 x = \sin 2x$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$. **2**

(b) Given that

- $\frac{dy}{dx} = 6\tan x\cos^2 x$, and
- $y = 3$ when $x = 0$,

express y in terms of x. **4**

a) $2\tan x \cos^2 x = \sin 2x$

$LHS = 2\tan x \cos^2 x$

$LHS = 2\left(\frac{\sin x}{\cos x}\right)\cos^2 x$ ✓

$LHS = 2\sin x \cos x$

$LHS = \sin 2x$ ✓

$LHS = RHS$ as required.

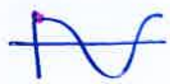
b) $y = \int 6\tan x\cos^2 x \, . \, dx$

$y = \int 3\sin 2x \, . \, dx$ ✓

$y = 3 \times -\frac{1}{2}\cos 2x + c$ ✓✓

$y = -\frac{3}{2}\cos 2x + c$

Point $(0, 3)$



$3 = -\frac{3}{2}\cos 2(0) + c$ $\cos 0 = 1$

$3 = -\frac{3}{2}(1) + c$

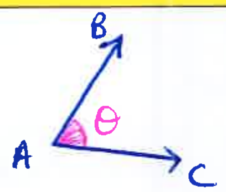
$c = 9/2$

↳ $y = -\frac{3}{2}\cos 2x + \frac{9}{2}$ ✓

SECTION 2 – PART A

11. (a) Given A (3, 1, 8), B (–2, 5, 1) and C (7, –6, 3), express $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in component form. **2**

(b) Hence calculate the size of angle BAC. **4**



a) $\overrightarrow{AB} = b - a$

$= \begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 8 \end{pmatrix}$

$\overrightarrow{AB} = \begin{pmatrix} -5 \\ 4 \\ -7 \end{pmatrix}$ ✓

$\overrightarrow{AC} = c - a$

$= \begin{pmatrix} 7 \\ -6 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 8 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} 4 \\ -7 \\ -5 \end{pmatrix}$ ✓

b) $\overrightarrow{AB}.\overrightarrow{AC} = (-5)(4) + (4)(-7) + (-7)(-5)$

$= -20 - 28 + 35$

$\overrightarrow{AB}.\overrightarrow{AC} = -13$ ✓

$\cos BAC = \dfrac{\overrightarrow{AB}.\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}$

$|\overrightarrow{AB}| = \sqrt{(-5)^2 + (4)^2 + (-7)^2}$

$= \sqrt{25 + 16 + 49}$

$|\overrightarrow{AB}| = \sqrt{90}$

$\cos BAC = \dfrac{-13}{\sqrt{90}\sqrt{90}}$ ✓

$|\overrightarrow{AC}| = \sqrt{(4)^2 + (-7)^2 + (-5)^2}$

$|\overrightarrow{AC}| = \sqrt{90}$ ✓

$\cos BAC = -\dfrac{13}{90}$

$BAC = \cos^{-1}\left(-{}^{13}/_{90}\right)$

$BAC = 98.3051...$

Angle $BAC = 98.3^\circ$ ✓ (1dp)

12. A sequence of real numbers is such that

- the terms of the sequence satisfy the recurrence relation $u_{n+1} = 9u_n - 440$
- $u_{n+1} > u_n$ for all values of n.

The difference between two particular terms, u_{k+1} and u_k, is 1000.

Determine, algebraically, the value of u_k. **3**

$U_{n+1} = 9U_n - 440$

$\therefore \; U_{k+1} = 9U_k - 440$

$U_{k+1} - U_k = 1000$

$U_{k+1} = U_k + 1000$

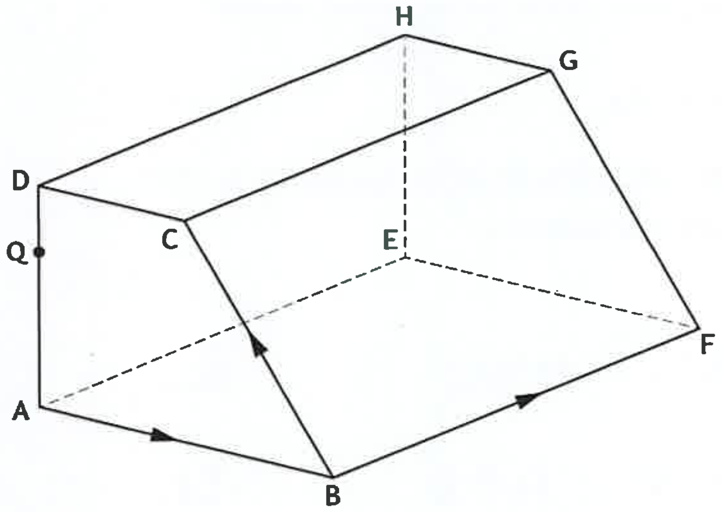
Let $U_{k+1} = U_{k+1}$

$9U_k - 440 = U_k + 1000$

$8U_k = 1440$

$U_k = 180$

13. ABCD,EFGH is a prism.



- $\overrightarrow{AB} = \begin{pmatrix} 8 \\ -4 \\ 6 \end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix} -7 \\ 5 \\ 3 \end{pmatrix}$ and $\overrightarrow{BF} = \begin{pmatrix} 7 \\ 11 \\ -2 \end{pmatrix}$.

$\overrightarrow{DC} = \frac{1}{2}\overrightarrow{AB}$

- $\overrightarrow{AB} = 2\overrightarrow{DC}$.

(a) Express $\overrightarrow{CF}$ in component form. **1**

(b) Hence, or otherwise, express $\overrightarrow{DF}$ in component form. **1**

(c) The point Q lies on the line AD.

Given that $\overrightarrow{QF} = \begin{pmatrix} 17 \\ 5 \\ 0 \end{pmatrix}$, find $\overrightarrow{QD}$. **2**

a) $\overrightarrow{CF} = \overrightarrow{CB} + \overrightarrow{BF}$

$\overrightarrow{CF} = -\overrightarrow{BC} + \overrightarrow{BF}$

$\overrightarrow{CF} = \begin{pmatrix} 7 \\ -5 \\ -3 \end{pmatrix} + \begin{pmatrix} 7 \\ 11 \\ -2 \end{pmatrix}$

$\overrightarrow{CF} = \begin{pmatrix} 14 \\ 6 \\ -5 \end{pmatrix}$ ✓

13) b) $\vec{DF} = \vec{DC} + \vec{CF}$

$\vec{DF} = \frac{1}{2}\vec{AB} + \vec{CF}$

$\vec{DF} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 14 \\ 6 \\ -5 \end{pmatrix}$

$\vec{DF} = \begin{pmatrix} 18 \\ 4 \\ -2 \end{pmatrix}$ ✓

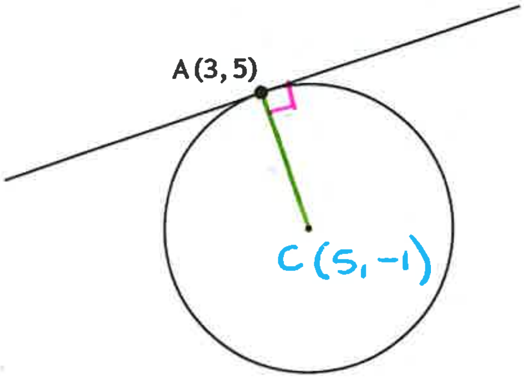
c) $\vec{QD} = \vec{QF} + \vec{FD}$ ✓

$\vec{QD} = \vec{QF} - \vec{DF}$

$\vec{QD} = \begin{pmatrix} 17 \\ 5 \\ 0 \end{pmatrix} - \begin{pmatrix} 18 \\ 4 \\ -2 \end{pmatrix}$

$\vec{QD} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ ✓

14. The point A (3, 5) lies on the circle with equation $x^2 + y^2 - 10x + 2y - 14 = 0$.



Find the equation of the tangent to the circle at A. **4**

Circle $x^2 + y^2 - 10x + 2y - 14 = 0$

$x^2 + y^2 + 2gx + 2fy + c = 0$

$2g = -10 \quad 2f = 2 \quad c = -14$

$g = -5 \quad f = 1$

Centre $(5, -1)$

Gradient of radius

$M_{rad} = \frac{-1-5}{5-3} = \frac{-6}{2} = -3$

Gradient of tangent

$M_{tan} = \frac{1}{3}$

Since $M_{rad} \times M_{tan} = -1$

Equation of tangent, $A(3,5)$

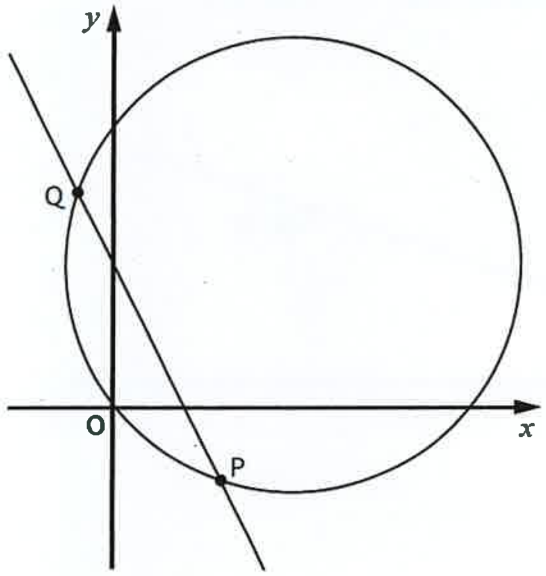
$y - b = m(x - a)$

$y - 5 = \frac{1}{3}(x - 3)$

$3y - 15 = x - 3$

$3y = x + 12$

15. The line $y=4-2x$ intersects the circle $x^2+y^2-10x-8y+1=0$ at the points P and Q.



Find the coordinates of the points of intersection. **4**

If $x^2 + y^2 - 10x - 8y + 1 = 0$

then $x^2 + (4-2x)^2 - 10x - 8(4-2x) + 1 = 0$ ✓

$x^2 + 16 - 16x + 4x^2 - 10x - 32 + 16x + 1 = 0$

$5x^2 - 10x - 15 = 0$ ✓

$5(x^2 - 2x - 3) = 0$

$5(x+1)(x-3) = 0$

$x + 1 = 0$ $\quad$ $x - 3 = 0$

$x = -1$ $\quad$ $x = 3$ ✓

$y = 4-2(-1)$ $\quad$ $y = 4-2(3)$

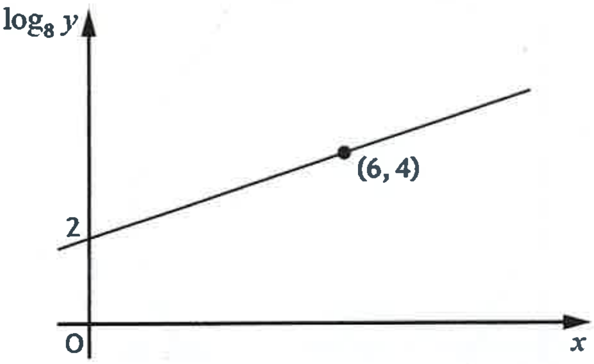
$y = 4+2$ $\quad$ $y = 4-6$

$y = 6$ $\quad$ $y = -2$

Points $Q(-1, 6)$ and $P(3, -2)$ ✓

16. Two variables, x and y, are connected by the equation $y = ab^x$.

The graph of $\log_8 y$ against x is a straight line as shown.



Find the values of a and b. **5**

Gradient (0,2) and (6,4)

$m = \frac{4-2}{6-0} = \frac{2}{6} = \frac{1}{3}$

y-intercept (0,2)

$\therefore c = 2$

Consider $Y = mX + c$

$\log_8 y = \frac{1}{3}x + 2$ ✓

$\log_8 y = \frac{1}{3}x \log_8 8 + 2\log_8 8$ ✓

$\log_8 y = \log_8 8^{1/3 x} + \log_8 8^2$ ✓

$\log_8 y = \log_8 2^x + \log_8 64$

$\log_8 y = \log_8 (64 \times 2^x)$ ✓

$y = 64 . 2^x$

where $a = 64$ and $b = 2$ ✓