National Qualifications

X847/76/11

Mathematics
Paper 1 (Non-calculator)

Duration — 1 hour 15 minutes

2021 PAPER 1 – WORKED SOLUTIONS

H Wallace

Total marks — 55

SECTION 1 — 44 marks

Attempt ALL questions.

SECTION 2 — 11 marks

Attempt EITHER Part A OR Part B.

1. Find the value of k for which the equation $kx^2 + 3x - 4 = 0$ has equal roots. 3

$ax^2 + bx + c = 0$

$a = k$

$b = 3$

$c = -4$

$b^2 - 4ac = 0$ for equal roots

$(3)^2 - 4(k)(-4) = 0$ ✓

$9 + 16k = 0$ ✓

$16k = -9$

$k = -\frac{9}{16}$ ✓

2. Given that $f(x) = (x^2+1)^5$, find $f'(1)$. 3

$f(x) = (x^2+1)^5$

$f'(x) = 5(x^2+1)^4$ ✓ $\times 2x$ ✓

$f'(x) = 10x(x^2+1)^4$

$f'(1) = 10(1).(1^2+1)^4$

$f'(1) = 10(2)^4$

$f'(1) = 10 \times 16$

$f'(1) = 160$ ✓

3. A function $f(x)$ is defined on $\mathbb{R}$, by

$$f(x)=\frac{x+3}{2}.$$

Find the inverse function, $f^{-1}(x)$. 3

Let $y = \dfrac{x+3}{2}$

$2y = x + 3$ ✓

$x = 2y - 3$ ✓

$\hookrightarrow$ $f^{-1}(x) = 2x - 3$ ✓

4. Determine whether the line passing through $(-4, 2)$ and $(2, -7)$ is perpendicular to the line with equation $3y = 2x + 9$. 3

Gradient between points

$m_1 = \dfrac{-7-2}{2-(-4)} = \dfrac{-9}{6} = -\dfrac{3}{2}$ ✓

Perpendicular gradient

$m_\perp = \dfrac{2}{3}$

Gradient of line

$3y = 2x + 9$

$y = \dfrac{2}{3}x + 3$

$y = mx + c$

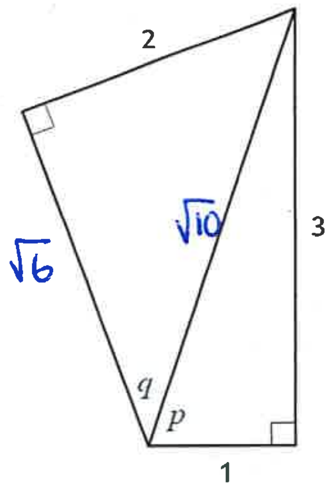
$m_2 = \dfrac{2}{3}$ ✓

Since $m_1 \times m_2 = -1$

$-\dfrac{3}{2} \times \dfrac{2}{3} = -1$

then the two lines are perpendicular. ✓

5. Two right-angled triangles are shown below.



$$\sqrt{3^2 + 1^2} = \sqrt{10}$$

$$\sqrt{(\sqrt{10})^2 - 2^2} = \sqrt{6}$$

(a) Determine the value of

(i) $\sin p$ **1**

(ii) $\cos q$. **2**

(b) Find the exact value of $\cos(p+q)$. **3**

a) $\sin p = \dfrac{3}{\sqrt{10}}$ and $\cos q = \dfrac{\sqrt{6}}{\sqrt{10}} = \dfrac{\sqrt{3}}{\sqrt{5}}$

b) $\sin q = \dfrac{2}{\sqrt{10}}$ $\cos p = \dfrac{1}{\sqrt{10}}$

$$\cos(p+q) = \cos p \cos q - \sin p \sin q$$

$$= \left(\frac{1}{\sqrt{10}}\right)\left(\frac{\sqrt{6}}{\sqrt{10}}\right) - \left(\frac{3}{\sqrt{10}}\right)\left(\frac{2}{\sqrt{10}}\right)$$

$$= \frac{\sqrt{6}}{10} - \frac{6}{10}$$

$$\cos(p+q) = \frac{\sqrt{6} - 6}{10}$$

6. Functions f and g are defined on $\mathbb{R}$ by

- $f(x) = 2x + 5$
- $g(x) = x^2 - 2x$.

(a) Find an expression for $f(g(x))$. **2**

(b) Find an expression for $g(f(x))$. **1**

(c) Express $g(f(x)) - f(g(x))$ in the form $a(x+b)^2 + c$. **4**

a) $f(g(x)) = f(x^2 - 2x)$

$= 2(x^2 - 2x) + 5$

$f(g(x)) = 2x^2 - 4x + 5$

b) $g(f(x)) = g(2x + 5)$

$= (2x+5)^2 - 2(2x+5)$

$= 4x^2 + 20x + 25 - 4x - 10$

$g(f(x)) = 4x^2 + 16x + 15$

c) $g(f(x)) - f(g(x)) = (4x^2 + 16x + 15) - (2x^2 - 4x + 5)$

$= 4x^2 + 16x + 15 - 2x^2 + 4x - 5$

$= [2x^2 + 20x] + 10$

$= 2[x^2 + 10x] + 10$

$= 2[(x+5)^2 - 25] + 10$

$= 2(x+5)^2 - 50 + 10$

$g(f(x)) - f(g(x)) = 2(x+5)^2 - 40$

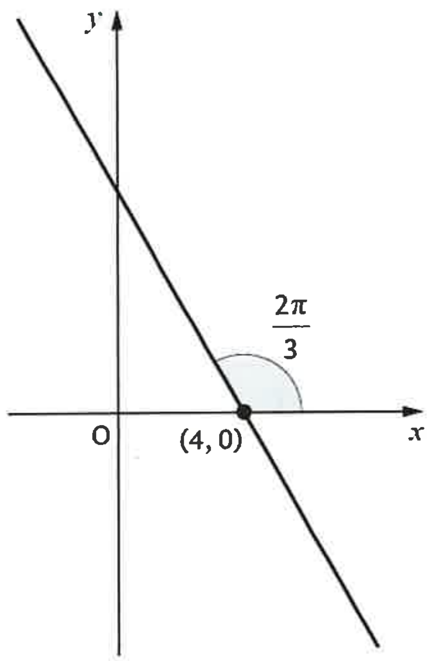
7. Find $\int 6\cos\left(3x+\frac{\pi}{4}\right)dx$. **2**

$= 6 \times \frac{1}{3}\sin\left(3x+\frac{\pi}{4}\right) + c$

$= 2\sin\left(3x+\frac{\pi}{4}\right) + c$

8. A line makes an angle of $\frac{2\pi}{3}$ with the positive direction of the x-axis.

It passes through the point $(4, 0)$.



$m = \tan\theta$

$m = \tan\frac{2\pi}{3}$

$m = \tan 120°$

$m = -\tan 60°$

$m = -\sqrt{3}$

Determine the equation of the line. **3**

Equation

$y - b = m(x-a)$

$m = -\sqrt{3}$

$y - 0 = -\sqrt{3}(x-4)$

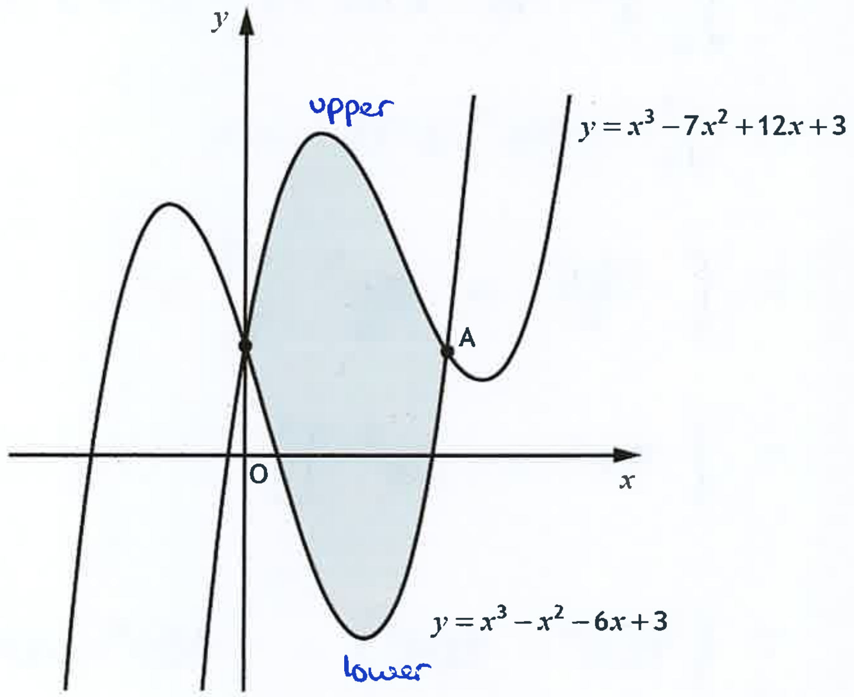
$a = 4$

$y = -\sqrt{3}x + 4\sqrt{3}$

$b = 0$

9. The diagram shows the curves with equations $y = x^3 - 7x^2 + 12x + 3$ and $y = x^3 - x^2 - 6x + 3$.

The curves intersect on the y-axis and at point A.



(a) Find the x-coordinate of A. **2**

(b) Calculate the shaded area. **5**

a) For points of intersection, let $y = y$

$$\cancel{x^3} - 7x^2 + 12x + \cancel{3} = \cancel{x^3} - x^2 - 6x + \cancel{3}$$

$$0 = 6x^2 - 18x$$

$$0 = 6x(x-3)$$

$6x = 0 \qquad x - 3 = 0$

$x = 0 \qquad x = 3$

The x-coordinate of A is 3.

9) b) Area $= \int_0^3 (x^3-7x^2+12x+3)-(x^3-x^2-6x+3).\,dx$

$= \int_0^3 x^3-7x^2+12x+3-x^3+x^2+6x-3.\,dx$

$= \int_0^3 18x - 6x^2.\,dx$

$= \left[\frac{18x^2}{2} - \frac{6x^3}{3}\right]_0^3$

$= \left[9x^2 - 2x^3\right]_0^3$

$= \left(9(3)^2 - 2(3)^3\right) - \left(9(0)^2 - 2(0)^3\right)$

$= (81 - 54) - 0$

Area $= 27$ units^2.

10. Factorise $6x^3 - 13x^2 + 4$ fully. 4

Try factors of 4 ... $\pm 1, \pm 2, \pm 4$

$$\begin{array}{r|rrrr} 1 & 6 & -13 & 0 & 4 \\ & \downarrow & 6 & -7 & -7 \\ \hline & 6 & -7 & -7 & \boxed{-3} \end{array}$$

$(x-1)$ is not a factor.

$$\begin{array}{r|rrrr} 2 & 6 & -13 & 0 & 4 \\ & \downarrow & 12 & -2 & -4 \\ \hline & 6 & -1 & -2 & \boxed{0} \end{array}$$

Since remainder is zero, then $x = 2$ is a root, and $(x-2)$ is a factor.

$\hookrightarrow$ $6x^3 - 13x^2 + 4$

$= (x-2)(6x^2 - x - 2)$

$= \underline{(x-2)(3x-2)(2x+1)}$

11. A function, f, defined on $\mathbb{R}$, is such that

- the maximum value of f is 8
- the maximum occurs when $x = 6$.

Max TP $(6, 8)$.

The function g is given by $g(x) = 2f(x) - 9$.

(a) State the maximum value of g. **1**

The function h is given by $h(x) = f(x-4) + 5$.

(b) (i) State the maximum value of h. **1**

(ii) State the value of x when the maximum value of h occurs. **1**

a) $g(x) = 2f(x) - 9$

double y-coords

down 9 units.

$\text{Max} = 2 \times 8 - 9$

$\text{Max} = 7$ ✓

b) $h(x) = f(x-4) + 5$

right 4 units

up 5 units.

i) $\text{Max} = 8 + 5$

$\text{Max} = 13$ ✓

ii) $x = 6 + 4$

$x = 10$ ✓

SECTION 2 – PART A

12. Points A, B, and C are collinear, with B dividing AC.

- A has coordinates $(4, 2, -5)$
- B has coordinates $(7, -4, 1)$
- $|\overrightarrow{BC}| = 6$

(a) (i) Find $|\overrightarrow{AB}|$. **2**

(ii) State the ratio in which B divides AC. **1**

(b) Determine the coordinates of C. **1**

a) i) $\overrightarrow{AB} = b - a$

$$= \begin{pmatrix} 7 \\ -4 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ -5 \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ -6 \\ 6 \end{pmatrix}$$

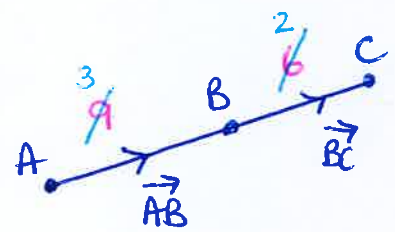
$$|\overrightarrow{AB}| = \sqrt{3^2 + (-6)^2 + 6^2}$$

$$= \sqrt{9 + 36 + 36}$$

$$= \sqrt{81}$$

$$|\overrightarrow{AB}| = 9 \text{ units}$$

ii)



Ratio 9 : 6

B divides AC in ratio 3:2

b) $c = b + \frac{2}{3}\overrightarrow{AB}$

$$c = \begin{pmatrix} 7 \\ -4 \\ 1 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 3 \\ -6 \\ 6 \end{pmatrix}$$

$$c = \begin{pmatrix} 7 \\ -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix}$$

$$c = \begin{pmatrix} 9 \\ -8 \\ 5 \end{pmatrix}$$

$\therefore$ Point C $(9, -8, 5)$

13. A sequence is generated by the recurrence relation $u_{n+1} = \frac{2}{3}u_n + 8$, $u_7 = 20$.

(a) Determine the value of u_5. **2**

This sequence approaches a limit as $n \to \infty$.

(b) Determine the limit of this sequence. **2**

a) $U_7 = \frac{2}{3}U_6 + 8$

$20 = \frac{2}{3}U_6 + 8$ ✓

$12 = \frac{2}{3}U_6$

$U_6 = 18$

$U_6 = \frac{2}{3}U_5 + 8$

$18 = \frac{2}{3}U_5 + 8$

$10 = \frac{2}{3}U_5$

$U_5 = 15$ ✓

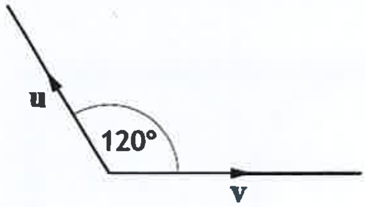
b) A limit exists since $-1 < \frac{2}{3} < 1$.

Limit, $L = \frac{8}{1 - 2/3}$ ✓ $= \frac{8}{1/3}$

Limit $= 24$. ✓

14. The angle between vectors $\mathbf{u}$ and $\mathbf{v}$ is 120°.

$|\mathbf{u}| = 4$ and $|\mathbf{v}| = 5$.



Calculate $\mathbf{u}.(\mathbf{u}+\mathbf{v})$. **3**

$$u.(u+v) = u.u + u.v$$

$$= |u|^2 + |u||v|\cos\theta°$$

$$= (4)^2 + (4)(5)\cos 120°$$

$$= 16 + 20(-\cos 60)$$

$$= 16 + 20\left(-\tfrac{1}{2}\right)$$

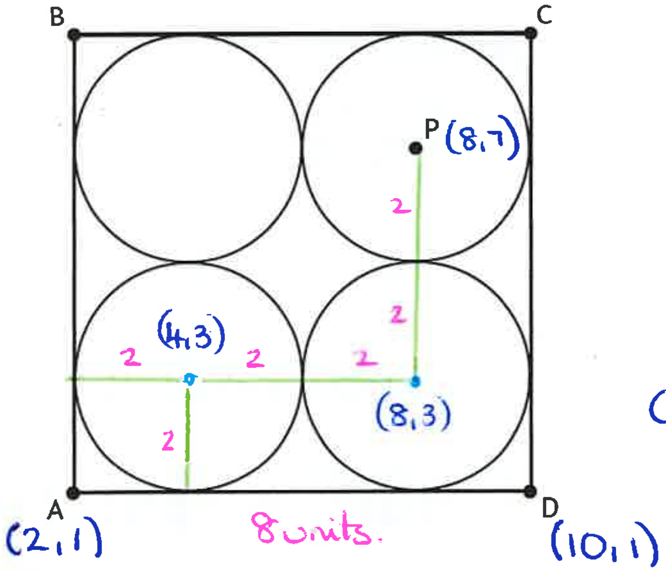
$$= 16 - 10$$

$$u.(u+v) = 6$$

SECTION 2 – PART B

15. ABCD is a square containing four congruent circles.

A is the point (2, 1), and D is the point (10, 1).



Square 8 by 8

Diameter = 4

Radius = 2

Centre P(8,7)

Determine the equation of the circle with centre P. **3**

Equation $(x-a)^2 + (y-b)^2 = r^2$

$(x-8)^2 + (y-7)^2 = 4$

16. Evaluate $\log_2 6 + \log_2 12 - 2\log_2 3$. **4**

$= \log_2 6 + \log_2 12 - \log_2 3^2$

$= \log_2 6 + \log_2 12 - \log_2 9$

$= \log_2 \left(\frac{6 \times 12}{9}\right)$

$= \log_2 \left(\frac{72}{9}\right)$

$= \log_2 8$

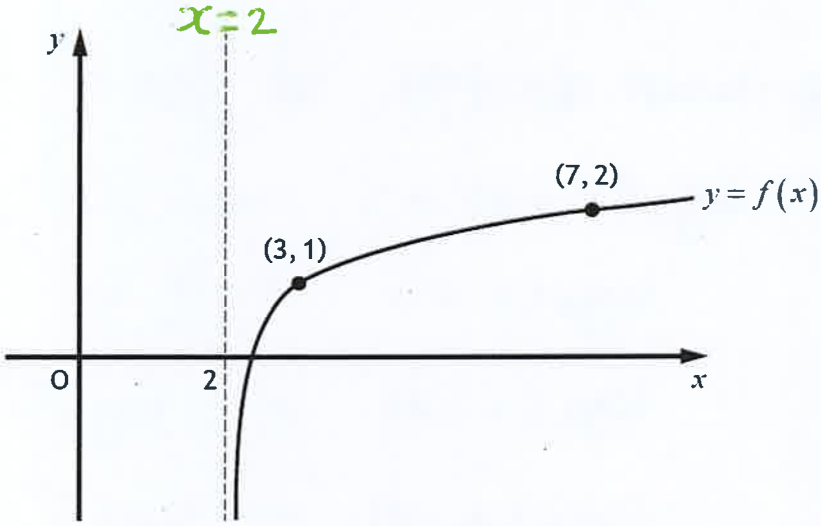
$= \log_2 2^3$

$= 3\log_2 2$

$= 3$

17. A logarithmic function, f, is defined for $x > 2$.

The diagram shows the graph of $y = f(x)$.



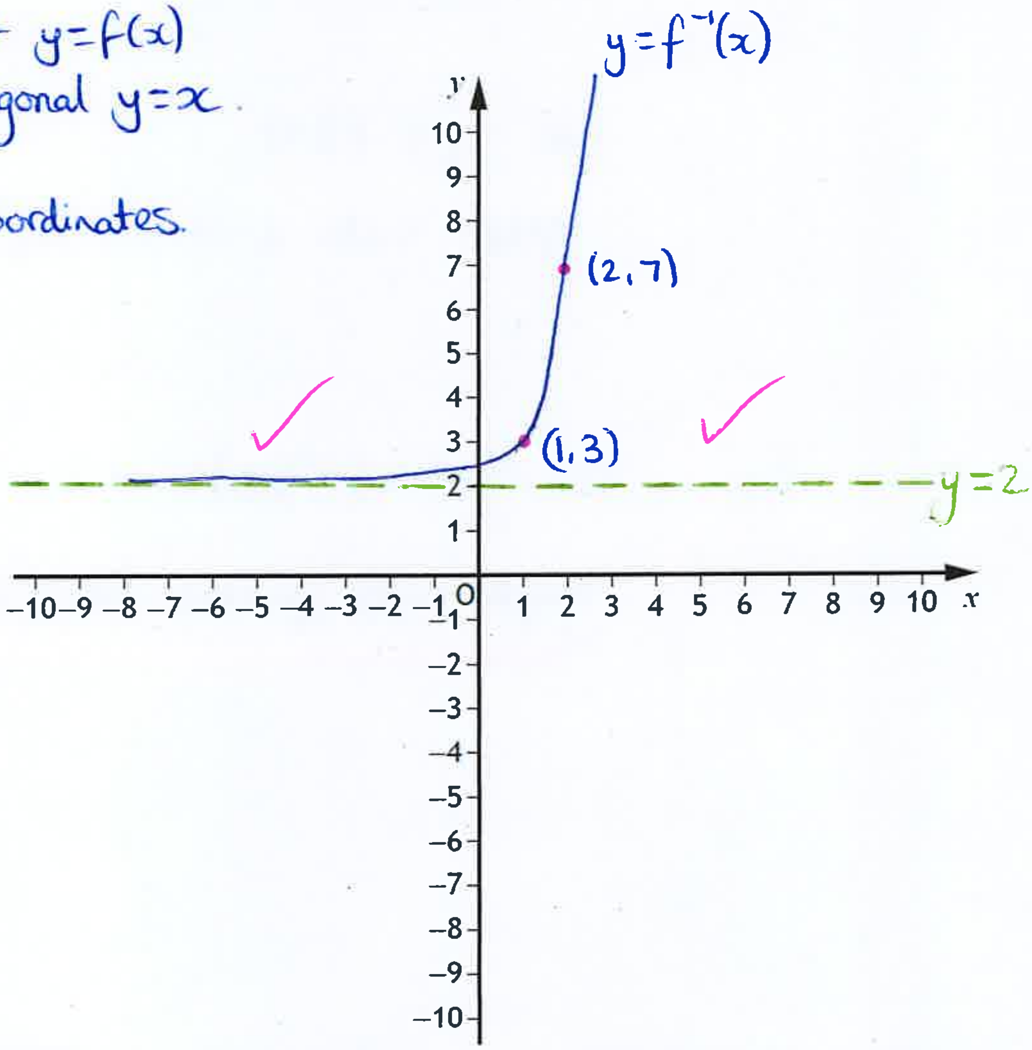
The inverse function, $f^{-1}(x)$, exists.

(a) On the diagram in your answer booklet, sketch the graph of the inverse function. **2**

(b) Given that $f(x) = \log_5(x-2)+1$, find the coordinates of the point where the graph of $f^{-1}(x)$ crosses the y-axis. **2**

Reflect $y=f(x)$
on diagonal $y=x$.

Flip coordinates.



17) b) $f(x) = \log_5(x-2) + 1$

For x-intercept of $f(x)$, let $y=0$

$$\log_5(x-2) + 1 = 0.$$

$$\log_5(x-2) = -1$$

$$\log_5(x-2) = -\log_5 5$$

$$\log_5(x-2) = \log_5 5^{-1}$$

$$x - 2 = 5^{-1}$$

$$x - 2 = \frac{1}{5}$$

$$x = 2\tfrac{1}{5}$$

For $y = f(x)$
graph cuts x-axis at $(2\tfrac{1}{5}, 0)$

↳ For $y = f^{-1}(x)$
graph cuts y-axis at $(0, 2\tfrac{1}{5})$