H

National
Qualifications
2019

X847/76/12

Mathematics
Paper 2

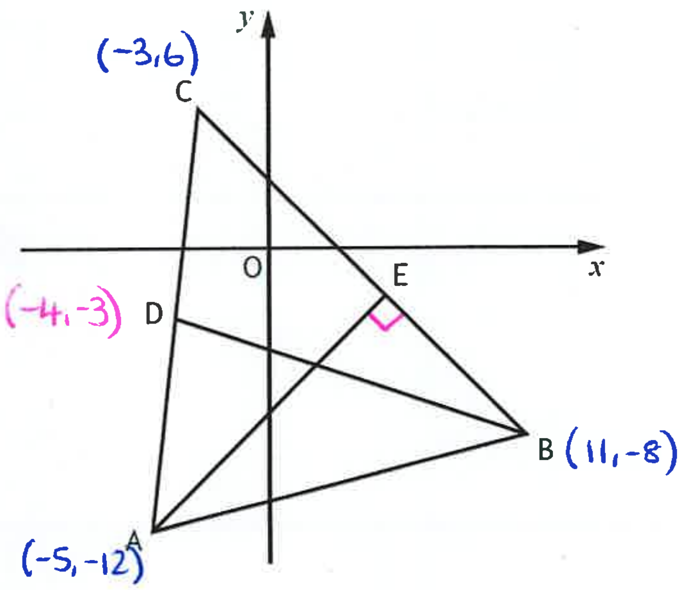
THURSDAY, 2 MAY
11:00 AM – 12:45 PM

80 marks

2019 PAPER 2 – WORKED SOLUTIONS

HWallace

1. Triangle ABC has vertices A(–5,–12), B(11,–8) and C(–3,6).



(a) Find the equation of the median BD. **3**

(b) Find the equation of the altitude AE. **3**

(c) Find the coordinates of the point of intersection of BD and AE. **2**

a) <u>Median BD</u>

Midpoint of AC

$D\left(\frac{-5+(-3)}{2}, \frac{-12+6}{2}\right)$

$D(-4, -3)$ ✓

Gradient of BD

$m_{BD} = \frac{-8-(-3)}{11-(-4)} = \frac{-5}{15} = -\frac{1}{3}$ ✓

Equation of BD

$y - b = m(x - a)$

$y + 3 = -\frac{1}{3}(x + 4)$

$3y + 9 = -x - 4$

$3y = -x - 13$ ✓

b) Altitude AE

Gradient of BC

$m_{BC} = \frac{-8-6}{11-(-3)} = \frac{-14}{14} = -1$

Gradient of AE

$m_{\perp} = 1$ since $m_1 \times m_2 = -1$

$m_{AE} = 1$

Equation of AE

$y - b = m(x - a)$

$y + 12 = 1(x + 5)$

$y + 12 = x + 5$

$y = x - 7$

c) Point of Intersection.

- $3y = -x - 13$

$y = x - 7$

- $3y = 3x - 21$.

Let $3y = 3y$

$3x - 21 = -x - 13$

$4x = 8$

$x = 2$

$\hookrightarrow y = 2 - 7 = -5$

Point $(2, -5)$

2. Find $\int\left(6\sqrt{x}-4x^{-3}+5\right)dx$.

prepare!

$$= \int 6x^{1/2} - 4x^{-3} + 5 \ . \quad dx$$

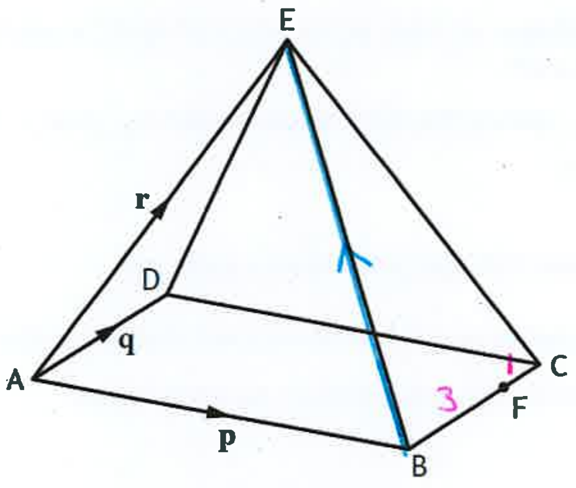
$$= \frac{6x^{3/2}}{3/2} - \frac{4x^{-2}}{-2} + 5x + c$$

$$= 4x^{3/2} + 2x^{-2} + 5x + c$$

$$= 4\sqrt{x^3} + \frac{2}{x^2} + 5x + c$$

3. E,ABCD is a rectangular based pyramid.

$\overrightarrow{AB} = \mathbf{p}$, $\overrightarrow{AD} = \mathbf{q}$ and $\overrightarrow{AE} = \mathbf{r}$.



(a) Express $\overrightarrow{BE}$ in terms of $\mathbf{p}$ and $\mathbf{r}$. **1**

Point F divides BC in the ratio 3:1.

(b) Express vector $\overrightarrow{EF}$ in terms of $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$. **2**

a) $\overrightarrow{BE} = \overrightarrow{BA} + \overrightarrow{AE}$

$\overrightarrow{BE} = -p + r$ ✓

$\overrightarrow{BE} = r - p$

b) $\overrightarrow{EF} = -\overrightarrow{BE} + \overrightarrow{BF}$ ✓

$\overrightarrow{EF} = -(r - p) + \frac{3}{4}q$

$= -r + p + \frac{3}{4}q$ ✓

$\overrightarrow{EF} = p + \frac{3}{4}q - r$

4. In a forest, the population of a species of mouse is falling by 2·7% each year.

To increase the population scientists plan to release 30 mice into the forest at the end of March each year.

(a) u_n is the estimated population of mice at the start of April, n years after the population was first estimated.

It is known that u_n and u_{n+1} satisfy the recurrence relation $u_{n+1} = au_n + b$.

State the values of a and b. **1**

The scientists continue to release this species of mouse each year.

(b) (i) Explain why the estimated population of mice will stabilise in the long term. **1**

(ii) Calculate the long term population to the nearest hundred. **2**

a) $U_{n+1} = aU_n + b$

2.7% fall each year
∴ 97.3% remains.
$a = 0.973$

$U_{n+1} = 0.973U_n + 30$

30 regularly added
$b = 30$

b) i) A limit will exist long term for this linear recurrence relation as $-1 < 0.973 < 1$.

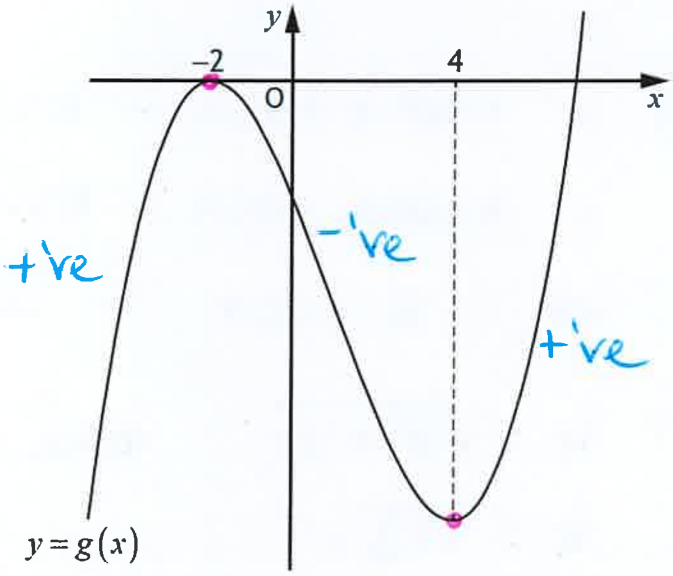
ii) Limit, $L = \dfrac{30}{1-0.973} = \dfrac{30}{0.027}$

$L = 1111.111...$

Long term, the population will be 1100

(nearest 100)

5. The diagram below shows the graph of a cubic function $y = g(x)$, with stationary points at $x = -2$ and $x = 4$.



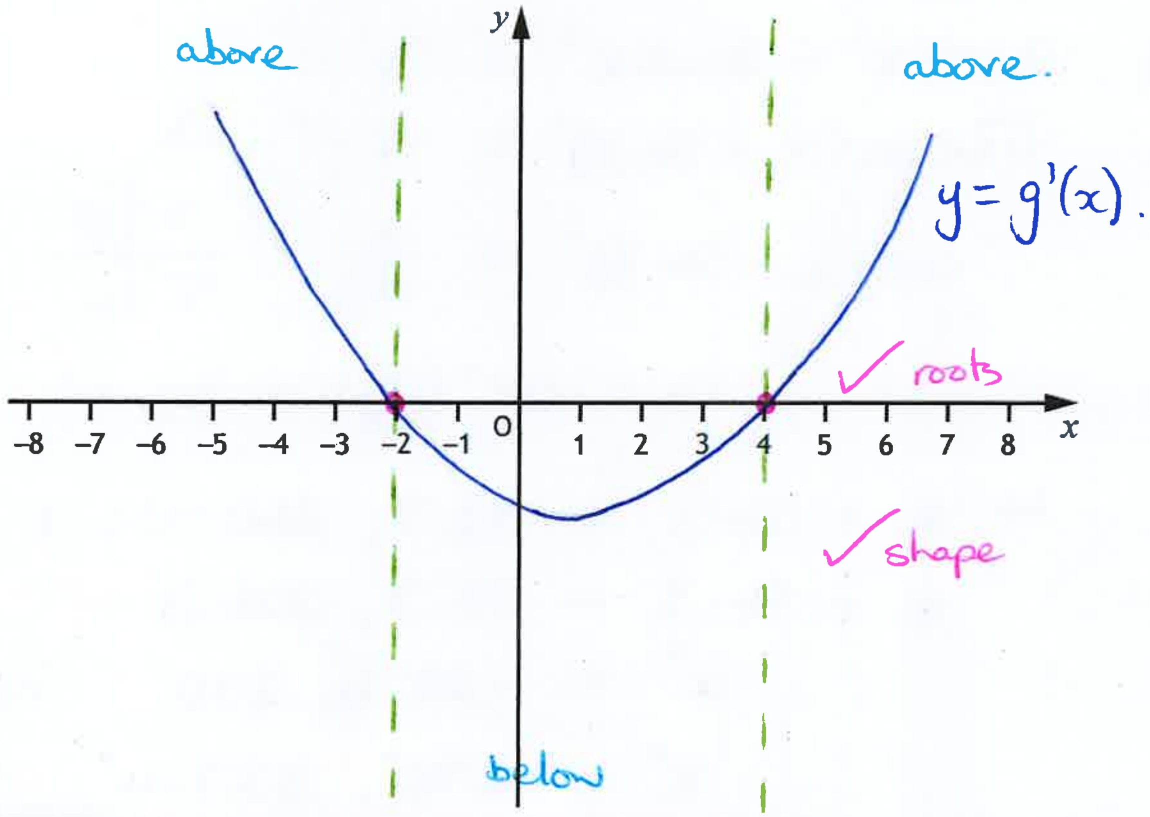
On the diagram in your answer booklet, sketch the graph of $y = g'(x)$. **2**

Stationary Points on $g(x)$ give roots on $g'(x)$

$\therefore$ $x = -2$ and $x = 4$.

Positive gradient on $g(x)$ means $g'(x) > 0$ (above)

Negative gradient on $g(x)$ means $g'(x) < 0$ (below)



6. (a) Express $2\cos x° - 3\sin x°$ in the form $k\cos(x+a)°$ where $k>0$ and $0 \le a < 360$. **4**

(b) Hence solve $2\cos x° - 3\sin x° = 3$ for $0 \le x < 360$. **3**

a) $k\cos(x+a) = k\cos x\cos a - k\sin x\sin a$ ✓

$= k\cos a . \cos x - k\sin a . \sin x$

$\rightarrow \quad 2 . \cos x - 3 . \sin x$

$k\sin a = 3$

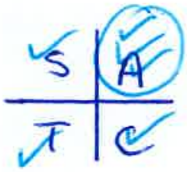
$k\cos a = 2$ ✓

$k = \sqrt{2^2 + 3^2}$

$k = \sqrt{13}$ ✓

$\tan a = \dfrac{k\sin a}{k\cos a}$

$\tan a = \dfrac{3}{2}$



$ra = \tan^{-1}\left(\frac{3}{2}\right) = 56.3°$

$a = 56.3°.$

$2\cos x° - 3\sin x° = \sqrt{13}\cos(x + 56.3)°$ ✓

b) $2\cos x° - 3\sin x° = 3$

$\sqrt{13}\cos(x+56.3)° = 3$ ✓

$\cos(x+56.3)° = \dfrac{3}{\sqrt{13}}$

$ra = \cos^{-1}\left(\frac{3}{\sqrt{13}}\right) = 33.7°.$

$\hookrightarrow \quad x + 56.3 = 33.7, \; 360 - 33.7$

$x + 56.3 = 33.7, \; 326.3$ ✓

$x° = \cancel{-22.6°}, \; 270°$ (+ 360)

$x° = 270°, \; 337.4°.$ ✓

7. (a) Express $-6x^2+24x-25$ in the form $p(x+q)^2+r$. **3**

(b) Given that $f(x)=-2x^3+12x^2-25x+9$,

show that $f(x)$ is strictly decreasing for all $x\in\mathbb{R}$. **3**

a) $[-6x^2 + 24x] - 25$

$-6[x^2 - 4x] - 25$

$-6[(x-2)^2 - 4] - 25$

$-6(x-2)^2 + 24 - 25$

$-6(x-2)^2 - 1$

b) $f(x) = -2x^3 + 12x^2 - 25x + 9$

$f'(x) = -6x^2 + 24x - 25$

$\therefore\ f'(x) = -6(x-2)^2 - 1$

Since $(x-2)^2 \geqslant 0$ for all values of x,

then $-6(x-2)^2 - 1 < 0$ for all values of x, and $f(x)$ is strictly decreasing.

8. A function, f, is given by $f(x)=\sqrt[3]{x}+8$.

The domain of f is $1 \leq x \leq 1000$, $x \in \mathbb{R}$.

The inverse function, f^{-1}, exists.

(a) Find $f^{-1}(x)$. **3**

(b) State the domain of f^{-1}. **1**

a) $f(x) = \sqrt[3]{x} + 8$

let $y = \sqrt[3]{x} + 8$

$y - 8 = \sqrt[3]{x}$ ✓

$(y-8)^3 = x$ ✓

↳ $f^{-1}(x) = (x-8)^3$ ✓

b) Consider the domain of f, $1 \leq x \leq 1000$

Now consider limits.

$f(1) = \sqrt[3]{1} + 8 = 9$

$f(1000) = \sqrt[3]{1000} + 8 = 18$

Domain of f^{-1} is $9 \leq x \leq 18$ ✓

9. Electricity on a spacecraft can be produced by a type of nuclear generator.

The electrical power produced by this generator can be modelled by

$$P_t = 120e^{-0.0079t}$$

where P_t is the electrical power produced, in watts, after t years.

(a) Determine the electrical power initially produced by the generator. **1**

(b) Calculate how long it takes for the electrical power produced by the generator to reduce by 15%. **4**

a) Initially, $t = 0$

$P_0 = 120e^{-0.0079(0)}$

$P_0 = 120e^0$

$P_0 = 120$ Watts initially.

b) 15% of 120 = 18

$P_t = 120 - 18 = 102$ Watts.

↳ $102 = 120e^{-0.0079t}$

$0.85 = e^{-0.0079t}$

$\ln 0.85 = \ln e^{-0.0079t}$

$\ln 0.85 = -0.0079t \ln e$

$t = \dfrac{\ln 0.85}{-0.0079}$

$t = 20.572016...$

It will take 20.572 years to reduce by 15%.

10. (a) Show that $(x+3)$ is a factor of $3x^4+10x^3+x^2-8x-6$. **2**

(b) Hence, or otherwise, factorise $3x^4+10x^3+x^2-8x-6$ fully. **5**

a) Check factor $(x+3)$

$$\begin{array}{r|rrrrr} -3 & 3 & 10 & 1 & -8 & -6 \\ & \downarrow & -9 & -3 & 6 & 6 \\ \hline & 3 & 1 & -2 & -2 & \boxed{0} \end{array}$$

Since the remainder is zero, then $x=-3$ is a root, and $(x+3)$ is a factor.

b) $3x^4 + 10x^3 + x^2 - 8x - 6$

$(x+3)(3x^3+x^2-2x-2)$

$$\begin{array}{r|rrrr} 1 & 3 & 1 & -2 & -2 \\ & \downarrow & 3 & 4 & 2 \\ \hline & 3 & 4 & 2 & \boxed{0} \end{array}$$

$\therefore (x-1)$ is also a factor

$\hookrightarrow (x+3)(x-1)(3x^2+4x+2)$

check discriminant

$a=3$ $\quad b^2-4ac = 4^2-4(3)(2)$

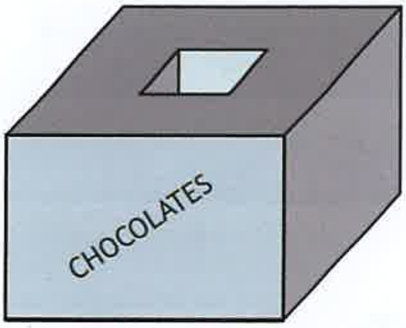
$b=4$ $\quad = 16-24$

$c=2$ $\quad b^2-4ac = -8$

Since $b^2-4ac<0$, there are no further factors.

$\hookrightarrow (x+3)(x-1)(3x^2+4x+2)$

11. A manufacturer of chocolates is launching a new product in novelty shaped cardboard boxes.



Consider Volume:

$V = 8x^2h$

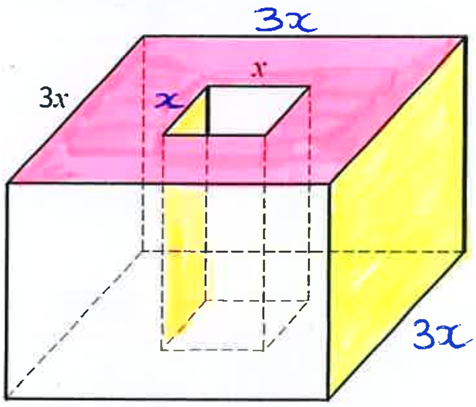
$8x^2h = 2000$

$h = \dfrac{2000}{8x^2}$ ✓

$h = \dfrac{250}{x^2}$

The box is a cuboid with a cuboid shaped tunnel through it.

- The height of the box is h centimetres
- The top of the box is a square of side $3x$ centimetres
- The end of the tunnel is a square of side x centimetres
- The volume of the box is 2000 cm^3



Consider Areas:

outside $= 3xh$

inside $= xh$

Top $= 9x^2 - x^2$

Top $= 8x^2$

(a) Show that the total surface area, A cm^2, of the box is given by

$$A = 16x^2 + \frac{4000}{x}.$$ **3**

(b) To minimise the cost of production, the surface area, A, of the box should be as small as possible.

Find the minimum value of A. **6**

a) Total Surface Area $= 4(3xh) + 4(xh) + 2(8x^2)$

$= 16x^2 + 16xh$ ✓

$A(x) = 16x^2 + 16x\left(\dfrac{250}{x^2}\right)$ ✓

$A = 16x^2 + \dfrac{4000}{x}$ as required.

11) b) $A(x) = 16x^2 + 4000x^{-1}$ ✓

$A'(x) = 32x - 4000x^{-2}$ ✓

$A'(x) = 32x - \dfrac{4000}{x^2}$

Stationary Points occur at $A'(x) = 0$

$$32x - \frac{4000}{x^2} = 0 \quad ✓$$

$$32x = \frac{4000}{x^2}$$

$$32x^3 = 4000$$

$$x^3 = 125$$

$$x = 5 \text{ cm} \quad ✓$$

Nature Table:

x	$\rightarrow$	5	$\rightarrow$
$A'(x)$	$-$	0	$+$
slope	\	—	/

$A'(4) = 32(4) - \dfrac{4000}{16} = -122$

$A'(6) = 32(6) - \dfrac{4000}{36} = 80\frac{8}{9}$

✓

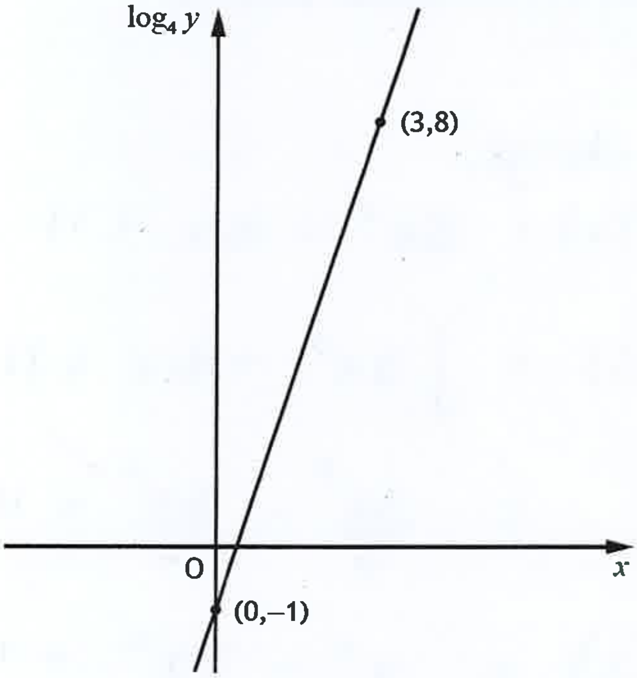
$\therefore$ A minimum area occurs when $x = 5$cm

$$A(5) = 16(5)^2 + \frac{4000}{5} = 1200$$

Minimum Area $= 1200 \text{ cm}^2$. ✓

12. Two variables, x and y, are connected by the equation $y = ab^x$.

The graph of $\log_4 y$ against x is a straight line as shown.



Find the values of a and b. **5**

Gradient

$m = \dfrac{8-(-1)}{3-0} = \dfrac{9}{3} = 3$

Y-intercept, $c = -1$

Consider $Y = mX + c$

$\log_4 y = 3x - 1$

$\log_4 y = 3x \log_4 4 - \log_4 4$

$\log_4 y = \log_4 4^{3x} - \log_4 4$

$\log_4 y = \log_4 \left(\dfrac{4^{3x}}{4}\right)$

$y = \dfrac{1}{4} \cdot 4^{3x}$

$y = \dfrac{1}{4} \cdot 64^{x}$

$a = \dfrac{1}{4}$ and $b = 64$

$y = a \cdot b^x$

13. For a function, f, defined on the set of real numbers, $\mathbb{R}$, it is known that

- the rate of change of f with respect to x is given by $3x^2 - 16x + 11$
- the graph with equation $y = f(x)$ crosses the x-axis at $(7,0)$.

Express $f(x)$ in terms of x. **5**

For rate of change,

then $f'(x) = 3x^2 - 16x + 11$

$\Rightarrow \quad f(x) = \int 3x^2 - 16x + 11 \, dx$

$= \frac{3x^3}{3} - \frac{16x^2}{2} + 11x + c$

$f(x) = x^3 - 8x^2 + 11x + c$

At $(7,0)$ then

$0 = (7)^3 - 8(7)^2 + 11(7) + c$

$0 = 343 - 392 + 77 + c$

$0 = 28 + c$

$c = -28$

$\Rightarrow \quad f(x) = x^3 - 8x^2 + 11x - 28$

14. The vectors **u** and **v** are such that

- $|\mathbf{u}| = 4$
- $|\mathbf{v}| = 5$
- $\mathbf{u}.(\mathbf{u}+\mathbf{v}) = 21$

Determine the size of the angle between the vectors **u** and **v**. **4**

$u.(u+v) = 21$

$u.u + u.v = 21$ ✓

$16 + u.v = 21$ ✓

$u.v = 5$

$u.u = |u|^2$

$= 4^2$

$u.u = 16$

$u.v = |u||v|\cos\theta$

$5 = (4)(5)\cos\theta$

$5 = 20\cos\theta$ ✓

$\cos\theta = \frac{5}{20}$

$\cos\theta = \frac{1}{4}$

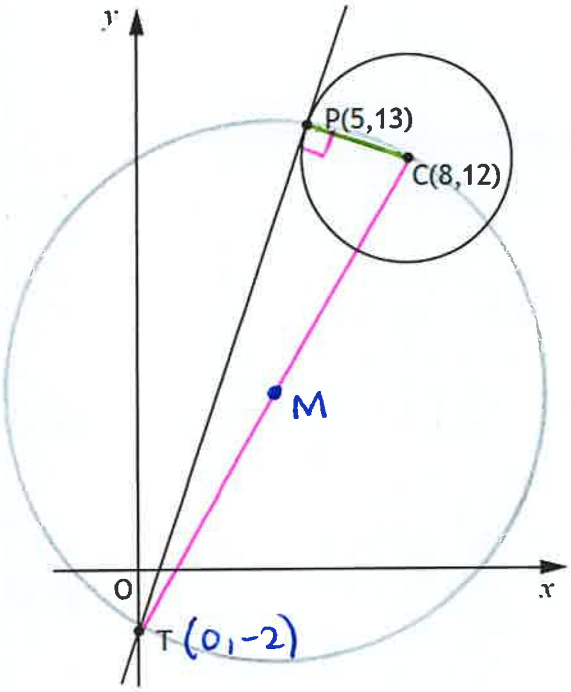
$ra = \cos^{-1}\left(\frac{1}{4}\right) = 75.5°$.

$\theta = 75.5°$.

Angle between u and v is 75.5°. ✓

15. A circle has centre C(8,12).

The point P(5,13) lies on the circle as shown.



(a) Find the equation of the tangent at P. **3**

The tangent from P meets the y-axis at the point T.

(b) (i) State the coordinates of T. **1**

(ii) Find the equation of the circle that passes through the points C, P and T. **3**

a) Gradient of radius

$m_{rad} = \frac{12-13}{8-5} = \frac{-1}{3}$ ✓

Gradient of tangent

$m_t = 3$ since $m_1 \times m_2 = -1$

$m_{tan} = 3$ ✓

Equation of tangent:

$y - b = m(x - a)$

$y - 13 = 3(x - 5)$

$y - 13 = 3x - 15$

$y = 3x - 2$ ✓

15) b) Y-intercept at T when $x = 0$

$$y = 3(0) - 2$$
$$y = -2$$

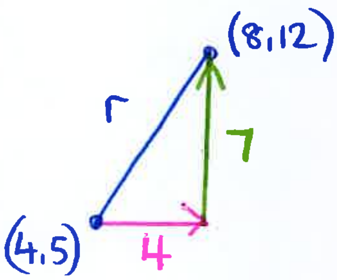
Point $T(0, -2)$

c) A circle passing C, P and T will have diameter CT, think right-angled triangle is a semi-circle.

Centre $M\left(\frac{0+8}{2}, \frac{-2+12}{2}\right)$

$M(4, 5)$

Radius MC



$$r^2 = 4^2 + 7^2$$
$$r^2 = 65$$
$$r = \sqrt{65}$$

Equation of circle

$$(x - a)^2 + (y - b)^2 = r^2$$
$$(x - 4)^2 + (y - 5)^2 = 65$$