National
Qualifications
2019

X847/76/11

Mathematics
Paper 1 (Non-calculator)

THURSDAY, 2 MAY
9:00 AM – 10:30 AM

70 marks

2019 PAPER 1 – WORKED SOLUTIONS

H Wallace

1. Find the x-coordinates of the stationary points on the curve with equation $y=\frac{1}{2}x^4-2x^3+6$. 4

$$\frac{dy}{dx} = 2x^3 - 6x^2$$

Stationary Points at $\frac{dy}{dx}=0$

$$2x^3 - 6x^2 = 0$$

$$2x^2(x-3) = 0$$

$2x^2 = 0 \qquad x-3=0$

$x = 0 \qquad x = 3$

Stationary Points at $x = 0, 3.$

2. The equation $x^2+(k-5)x+1=0$ has equal roots.

Determine the possible values of k. 3

$$x^2 + (k-5)x + 1 = 0$$

$$ax^2 + bx + c = 0$$

$a=1$

$b=(k-5)$

$c=1$

$b^2 - 4ac = 0$ for equal roots

$$(k-5)^2 - 4(1)(1) = 0$$

$$k^2 - 10k + 25 - 4 = 0$$

$$k^2 - 10k + 21 = 0$$

$$(k-3)(k-7) = 0$$

$k=3$ or $k=7$

3. Circle C_1 has equation $x^2+y^2-6x-2y-26=0$.

Circle C_2 has centre (4,–2).

The radius of C_2 is equal to the radius of C_1.

Find the equation of circle C_2. **2**

C_1: $x^2+y^2-6x-2y-26=0$

$x^2+y^2+2gx+2fy+c=0$

$2g=-6 \quad 2f=-2 \quad c=-26$

$g=-3 \quad f=-1$

Radius $r_1 = \sqrt{(-3)^2+(-1)^2-(-26)}$

$r_1 = \sqrt{9+1+26}$

$r_1 = \sqrt{36} = 6$

Equation of circle C_2

$(x-a)^2+(y-b)^2=r^2$

$(x-4)^2+(y+2)^2=36$

4. A sequence is generated by the recurrence relation

$$u_{n+1}=mu_n+c,$$

where the first three terms of the sequence are 6, 9 and 11.

(a) Find the values of m and c. $U_1=6 \quad U_2=9 \quad U_3=11$ **3**

(b) Hence, calculate the fourth term of the sequence. **1**

a) $U_2 = mU_1 + c$ $\qquad U_3 = mU_2 + c$

$9 = 6m + c$ $\qquad 11 = 9m + c$

$c = 9 - 6m$ $\qquad c = 11 - 9m$

Let $c = c$

$9 - 6m = 11 - 9m$

$3m = 2$

$m = \frac{2}{3}$

$c = 9 - 6\left(\frac{2}{3}\right)$

$c = 9 - 4$

$c = 5$

b) $U_{n+1} = \frac{2}{3}U_n + 5$

$U_4 = \frac{2}{3}(11) + 5$

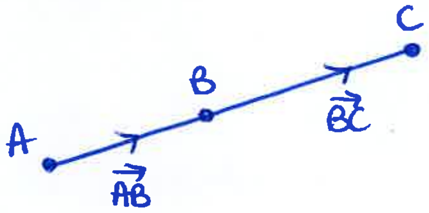
$U_4 = \frac{22}{3} + 5$

$U_4 = \frac{37}{3}$ or $12\frac{1}{3}$

5. (a) Show that the points A(1,5,–3), B(4,–1,0) and C(8,–9,4) are collinear. **3**

(b) State the ratio in which B divides AC. **1**

a)



$\vec{AB} = b - a$

$= \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$

$\vec{BC} = c - b$

$= \begin{pmatrix} 8 \\ -9 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}$

$\vec{AB} = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix}$ ✓

$\vec{BC} = \begin{pmatrix} 4 \\ -8 \\ 4 \end{pmatrix}$

$\vec{AB} = 3\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$\vec{BC} = 4\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Since $\dfrac{\vec{AB}}{\vec{BC}} = \dfrac{3}{4}$, ✓ then $4\vec{AB} = 3\vec{BC}$ and AB is parallel to BC. By sharing a common point B, then A, B and C ✓ are collinear.

b) B divides AC in the ratio 3 : 4. ✓

6. Given that $y=\dfrac{1}{(1-3x)^5}$, $x\neq\dfrac{1}{3}$, find $\dfrac{dy}{dx}$. **3**

Prepare!

$$y = (1-3x)^{-5}$$

$$\frac{dy}{dx} = -5(1-3x)^{-6} \times (-3)$$

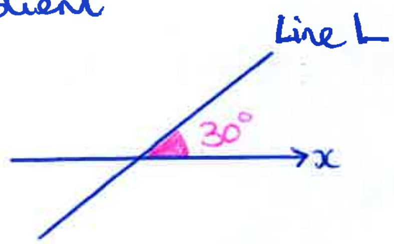
$$\frac{dy}{dx} = 15(1-3x)^{-6}$$

$$\frac{dy}{dx} = \frac{15}{(1-3x)^6}$$

7. The line, L, makes an angle of 30° with the positive direction of the x-axis.

Find the equation of the line perpendicular to L, passing through $(0,-4)$. **4**

Gradient

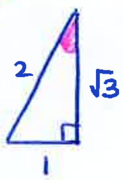


point $(0,-4)$

y-intercept, $c=-4$

$m = \tan 30$

$m = \dfrac{1}{\sqrt{3}}$



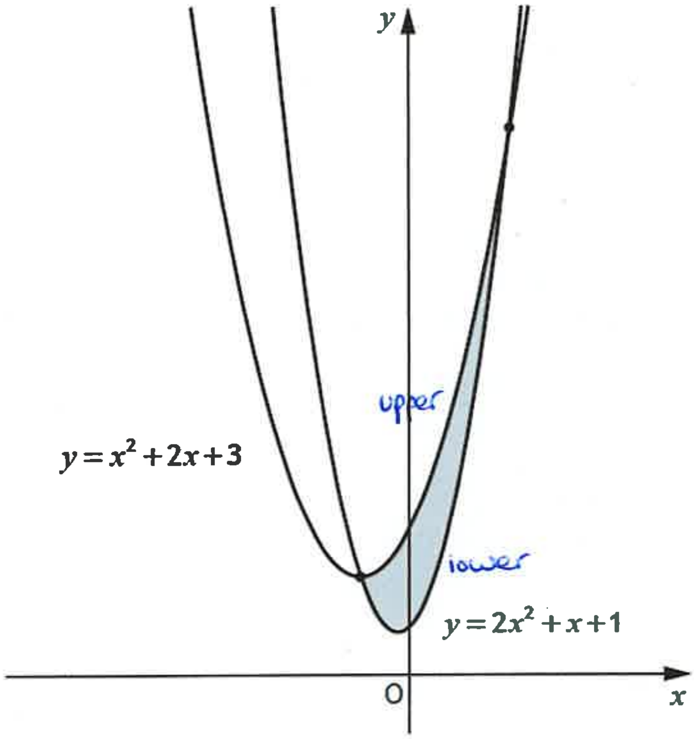
$m_{\perp} = -\sqrt{3}$

Equation:

$$y = mx + c$$

$$y = -\sqrt{3}x - 4$$

8. The graphs of $y=x^2+2x+3$ and $y=2x^2+x+1$ are shown below.



The graphs intersect at the points where $x=-1$ and $x=2$.

(a) Express the shaded area, enclosed between the curves, as an integral. 1

(b) Evaluate the shaded area. 3

a) Area $= \int_{-1}^{2} (x^2+2x+3) - (2x^2+x+1) \, dx$

b) Area $= \int_{-1}^{2} x^2+2x+3-2x^2-x-1 \, dx$

$= \int_{-1}^{2} 2+x-x^2 \, dx$

$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^{2}$

$= \left(2(2) + \frac{(2)^2}{2} - \frac{(2)^3}{3} \right) - \left(2(-1) + \frac{(-1)^2}{2} - \frac{(-1)^3}{3} \right)$

$= \left(4 + 2 - \frac{8}{3} \right) - \left(-2 + \frac{1}{2} + \frac{1}{3} \right)$

$= 6 - \frac{8}{3} + 2 - \frac{1}{2} - \frac{1}{3}$

$= 8 - \frac{9}{3} - \frac{1}{2}$

Area $= 4\frac{1}{2}$ units^2.

9. Vectors **u** and **v** have components $\begin{pmatrix} p \\ -2 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 2p+16 \\ -3 \\ 6 \end{pmatrix}$, $p \in \mathbb{R}$.

(a) (i) Find an expression for **u.v**. 1

(ii) Determine the values of p for which **u** and **v** are perpendicular. 3

(b) Determine the value of p for which **u** and **v** are parallel. 2

a) i) $u.v = p(2p+16) + (-2)(-3) + (4)(6)$

$= 2p^2 + 16p + 6 + 24$

$u.v = 2p^2 + 16p + 30$ ✓

ii) If $\underline{u}$ and $\underline{v}$ are perpendicular, then $u.v = 0$.

$\therefore \quad 2p^2 + 16p + 30 = 0$ ✓

$2(p^2 + 8p + 15) = 0$

$2(p+3)(p+5) = 0$ ✓

$p + 3 = 0 \qquad p + 5 = 0$

$p = -3$ and $p = -5$ ✓

b) If parallel then $3u = 2v$ by comparing components

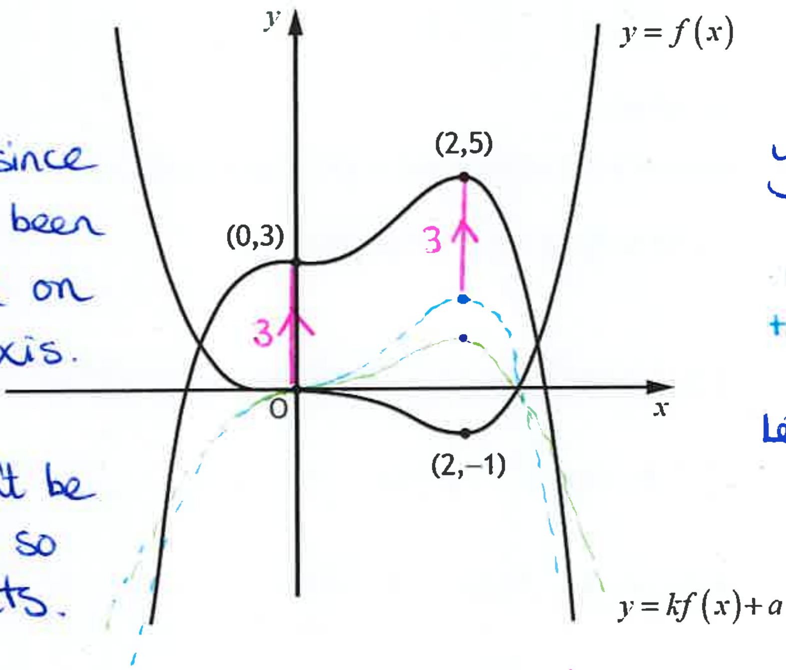
$\therefore \quad 3(p) = 2(2p+16)$ ✓

$3p = 4p + 32$

$p = -32$ ✓

10. The diagram shows the graphs with equations $y = f(x)$ and $y = kf(x) + a$.

k < 0 since f(x) has been reflected on the x-axis.



$y = kf(x) + a$

reflect + stretch   up 3

(0,0) can't be stretched, so up 3 units.

Let $k(-1) + 3 = 5$

$-k + 3 = 5$

$-k = 2$

$k = -2$

(a) State the value of a. **1**

$a = 3$

(b) Find the value of k. **1**

$k = -2$

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11. Evaluate $\int_0^{\frac{\pi}{9}} \cos\left(3x - \frac{\pi}{6}\right) dx$. **4**

$$= \left[\frac{1}{3}\sin\left(3x - \frac{\pi}{6}\right)\right]_0^{\pi/9}$$

$$= \frac{1}{3}\sin\left(\frac{3\pi}{9} - \frac{\pi}{6}\right) - \frac{1}{3}\sin\left(0 - \frac{\pi}{6}\right)$$

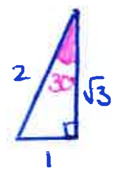
$$= \frac{1}{3}\sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right) - \frac{1}{3}\sin\left(-\frac{\pi}{6}\right)$$

$$= \frac{1}{3}\sin\left(\frac{\pi}{6}\right) - \frac{1}{3}\sin\left(-\frac{\pi}{6}\right)$$

$$= \frac{1}{3}\left(\frac{1}{2}\right) - \frac{1}{3}\left(-\frac{1}{2}\right)$$

$$= \frac{1}{6} + \frac{1}{6}$$

$$= \frac{1}{3}$$



$\sin 30 = \frac{1}{2}$

$\sin(-30) = -\frac{1}{2}$

12. Functions f and g are defined by

- $f(x)=\dfrac{1}{\sqrt{x}}$, where $x>0$
- $g(x)=5-x$, where $x\in\mathbb{R}$.

(a) Determine an expression for $f(g(x))$. **2**

(b) State the range of values of x for which $f(g(x))$ is undefined. **1**

a) $f(g(x)) = f(5-x)$ ✓

$$f(g(x)) = \frac{1}{\sqrt{5-x}}$$ ✓

b) Restrictions:

- We cannot divide by zero
- We cannot square root a negative

$\therefore \quad 5-x > 0$

$5 > x$

$f(g(x))$ is defined when $x < 5$.

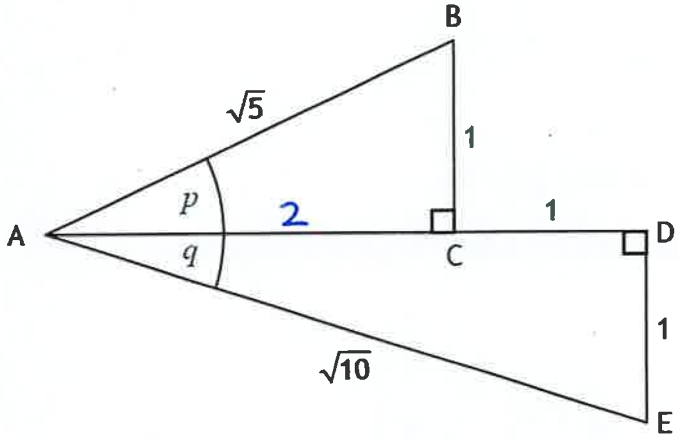
$\hookrightarrow$ undefined when $x \geqslant 5$ ✓

13. Triangles ABC and ADE are both right angled.

Angles p and q are as shown in the diagram.

$AC = \sqrt{(\sqrt{5})^2 - 1^2}$

$AC = 2$



(a) Determine the value of

(i) $\cos p$ **1**

(ii) $\cos q$. **1**

(b) Hence determine the value of $\sin(p+q)$. **3**

a) $\cos p = \dfrac{2}{\sqrt{5}}$ ✓ $\cos q = \dfrac{3}{\sqrt{10}}$ ✓

b) $\sin p = \dfrac{1}{\sqrt{5}}$ $\sin q = \dfrac{1}{\sqrt{10}}$

$$\sin(p+q) = \sin p \cos q + \cos p \sin q$$ ✓

$$= \left(\frac{1}{\sqrt{5}}\right)\left(\frac{3}{\sqrt{10}}\right) + \left(\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{10}}\right)$$ ✓

$$= \frac{3}{\sqrt{50}} + \frac{2}{\sqrt{50}}$$

$$= \frac{5}{5\sqrt{2}}$$

$$\sin(p+q) = \frac{1}{\sqrt{2}}$$ ✓

14. (a) Evaluate $\log_{10}4+2\log_{10}5$. 3

(b) Solve $\log_2(7x-2)-\log_2 3=5$. $x\geq 1$. 3

a) $\log_{10}4 + 2\log_{10}5$

$= \log_{10}4 + \log_{10}5^2$ ✓

$= \log_{10}(4 \times 25)$ ✓

$= \log_{10}100$

$= \log_{10}10^2$

$= 2\log_{10}10$

$= 2$ ✓

b) $\log_2(7x-2) - \log_2 3 = 5$

$\log_2\left(\frac{7x-2}{3}\right) = 5\log_2 2$ ✓

$\log_2\left(\frac{7x-2}{3}\right) = \log_2 2^5$

$\frac{7x-2}{3} = 32$ ✓

$7x-2 = 96$

$7x = 98$

$x = 14$ ✓

15. (a) Solve the equation $\sin 2x° + 6\cos x° = 0$ for $0 \le x < 360$. **4**

(b) Hence solve $\sin 4x° + 6\cos 2x° = 0$ for $0 \le x < 360$. **1**

a) $\sin 2x° + 6\cos x° = 0$

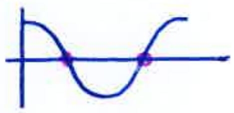
$2\sin x \cos x + 6\cos x = 0$

$2\cos x\,(\sin x + 3) = 0$

$2\cos x = 0$ $\qquad$ $\sin x + 3 = 0$

$\cos x = 0$ $\qquad$ ~~$\sin x = -3$~~

No solution.



$x° = 90°, 270°$

b) Consider $\sin 4x + 6\cos 2x = 0$

$\therefore \; 2x = 90, 270$

$x = 45, 135$ first wave

+180 $\quad$ +180

$x° = 45°, 135°, 225°, 315°$

16. The point P has coordinates $(4,k)$.

C is the centre of the circle with equation $(x-1)^2+(y+2)^2=25$.

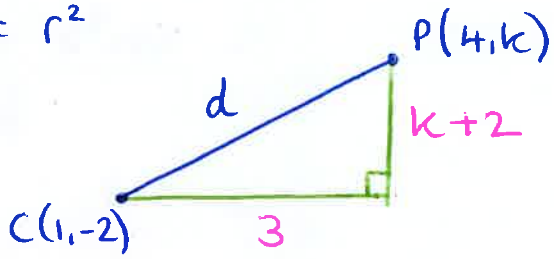
(a) Show that the distance between the points P and C is given by $\sqrt{k^2+4k+13}$. **2**

(b) Hence, or otherwise, find the range of values of k such that P lies outside the circle. **4**

a) Circle $(x-1)^2+(y+2)^2 = 25$

$(x-a)^2+(y-b)^2 = r^2$

Centre $(1,-2)$ ✓



Distance, $d = \sqrt{(3)^2+(k+2)^2}$

$d = \sqrt{9+k^2+4k+4}$ ✓

$d = \sqrt{k^2+4k+13}$ as required.

b) Radius of circle, $r=5$

If P lies outside the circle, then $d > r$

$\sqrt{k^2+4k+13} > 5$ ✓

$k^2+4k+13 > 25$

$k^2+4k-12 > 0$ ✓

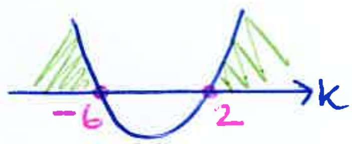
Consider roots:

$k^2+4k-12=0$

$(k-2)(k+6)=0$

$k=2 \qquad k=-6$ ✓

Sketch:



$k<-6$ or $k>2$ ✓

17. (a) Express $(\sin x - \cos x)^2$ in the form $p + q\sin rx$ where p, q and r are integers. **3**

(b) Hence, find $\int (\sin x - \cos x)^2\, dx$. **2**

a) $(\sin x - \cos x)^2 = (\sin x - \cos x)(\sin x - \cos x)$

$= \sin^2 x - \sin x\cos x - \sin x \cos x + \cos^2 x$

$= \underbrace{\sin^2 x + \cos^2 x}_{\downarrow} - \underbrace{2\sin x\cos x}_{\downarrow}$

$= 1 - \sin 2x$

b) $\int (\sin x - \cos x)^2 .\, dx$

$= \int 1 - \sin 2x \,.\, dx$

$= x + \frac{1}{2}\cos 2x + c$