

National
Qualifications
2018

X747/76/12

**Mathematics
Paper 2**

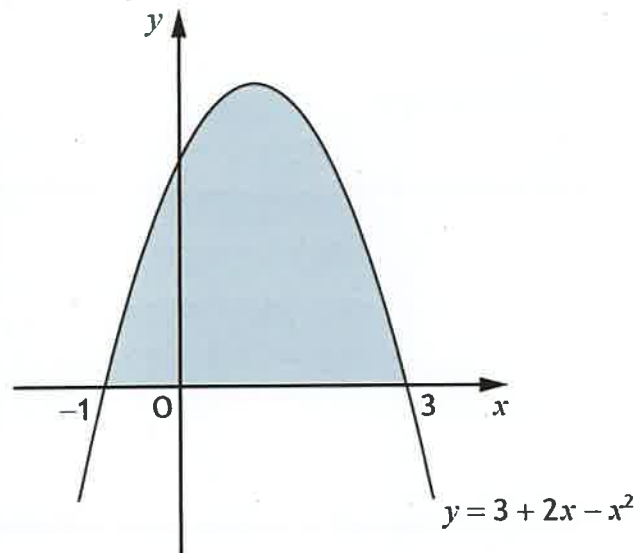
THURSDAY, 3 MAY
10:30 AM – 12:00 NOON

70 marks

2018 PAPER 2 - WORKED SOLUTIONS

H Wallace

1. The diagram shows the curve with equation $y = 3 + 2x - x^2$.



Calculate the shaded area.

4

$$\text{Area} = \int_{-1}^3 3 + 2x - x^2 \, dx \quad \checkmark$$

$$= \left[3x + \frac{2x^2}{2} - \frac{x^3}{3} \right]_{-1}^3 \quad \checkmark$$

$$= \left[3x + x^2 - \frac{x^3}{3} \right]_{-1}^3$$

$$= \left(3(3) + (3)^2 - \frac{(3)^3}{3} \right) - \left(3(-1) + (-1)^2 - \frac{(-1)^3}{3} \right) \quad \checkmark$$

$$= (9 + 9 - 9) - \left(-3 + 1 + \frac{1}{3} \right)$$

$$= 9 - \left(\frac{1}{3} - 2 \right)$$

$$= 11 - \frac{1}{3}$$

$$\text{Area} = 10\frac{2}{3} \text{ units}^2 \quad \checkmark$$

2. Vectors u and v are defined by $u = \begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix}$ and $v = \begin{pmatrix} -7 \\ 8 \\ 5 \end{pmatrix}$.

(a) Find $u \cdot v$.

1

(b) Calculate the acute angle between u and v .

4

$$\begin{aligned} a) \quad u \cdot v &= (-1)(-7) + (4)(8) + (-3)(5) \\ &= 7 + 32 - 15 \end{aligned}$$

$$u \cdot v = 24 \quad \checkmark$$



$$b) \quad \cos \theta = \frac{u \cdot v}{|u||v|}$$

$$|u| = \sqrt{(-1)^2 + (4)^2 + (-3)^2}$$

$$|u| = \sqrt{26} \quad \checkmark$$

$$\cos \theta = \frac{24}{\sqrt{26}\sqrt{138}} \quad \checkmark$$

$$|v| = \sqrt{(-7)^2 + (8)^2 + (5)^2}$$

$$|v| = \sqrt{138} \quad \checkmark$$

$$\theta = \cos^{-1} \left(\frac{24}{\sqrt{26}\sqrt{138}} \right)$$

$$\theta = 66.3800\dots$$

The angle between u and v is 66.4° (1dp).

3. A function, f , is defined on the set of real numbers by $f(x) = x^3 - 7x - 6$.

Determine whether f is increasing or decreasing when $x = 2$.

3

$$f(x) = x^3 - 7x - 6$$

$$f'(x) = 3x^2 - 7 \quad \checkmark$$

$$\text{At } x = 2$$

$$f'(2) = 3(2)^2 - 7$$

$$f'(2) = 12 - 7$$

$$f'(2) = 5 \quad \checkmark$$

Since $f'(2) > 0$, then f is increasing. \checkmark

4. Express $-3x^2 - 6x + 7$ in the form $a(x+b)^2 + c$.

3

$$[-3x^2 - 6x] + 7$$

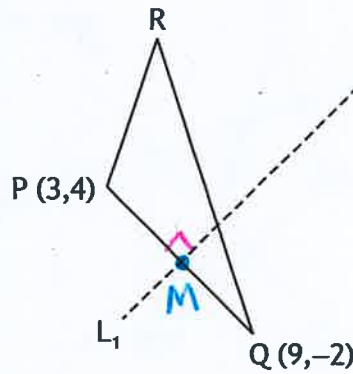
$$-3[x^2 + 2x] + 7 \quad \checkmark$$

$$-3[(x+1)^2 - 1] + 7 \quad \checkmark$$

$$-3(x+1)^2 + 3 + 7$$

$$\underline{\underline{-3(x+1)^2 + 10}} \quad \checkmark$$

5. PQR is a triangle with P(3,4) and Q(9,-2).



(a) Find the equation of L_1 , the perpendicular bisector of PQ.

3

Midpoint of PQ

$$M \left(\frac{3+9}{2}, \frac{4+(-2)}{2} \right)$$

$$M(6, 1) \checkmark$$

Gradient of PQ

$$m_{PQ} = \frac{-2-4}{9-3} = \frac{-6}{6} = -1$$

Gradient of L_1

$$m_{L_1} = 1 \checkmark \text{ since } m_1 \times m_2 = -1$$

Equation of L_1

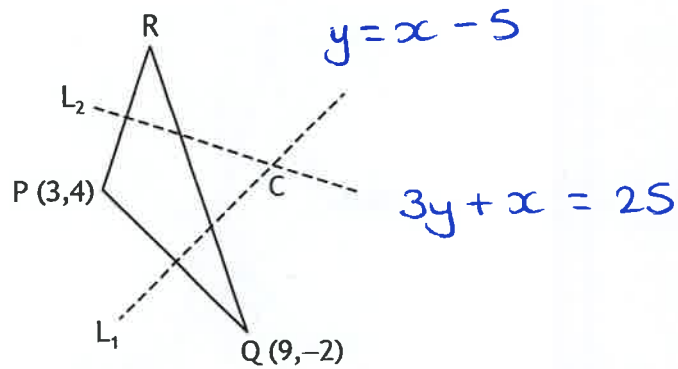
$$y - b = m(x - a)$$

$$y - 1 = 1(x - 6)$$

$$y - 1 = x - 6$$

$$\underline{\underline{y = x - 5}} \checkmark$$

The equation of L_2 , the perpendicular bisector of PR is $3y + x = 25$.



(b) Calculate the coordinates of C, the point of intersection of L_1 and L_2 .

2

$$y = x - 5$$

$$3y = 3x - 15$$

and $3y = -x + 25$

$$\text{Let } 3y = 3y$$

$$3x - 15 = -x + 25$$

$$4x = 40$$

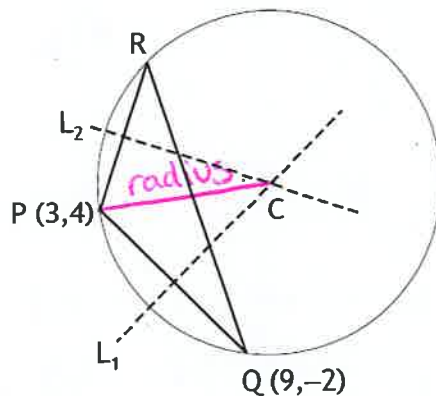
$$x = 10 \checkmark$$

$$y = 10 - 5$$

$$y = 5$$

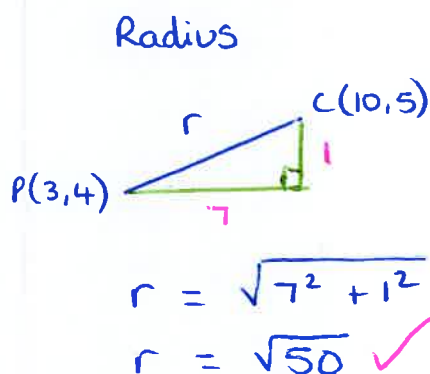
Point C (10, 5) ✓

C is the centre of the circle which passes through the vertices of triangle PQR.



(c) Determine the equation of this circle.

2



Equation of circle

$$(x-a)^2 + (y-b)^2 = r^2$$

$$\underline{\underline{(x-10)^2 + (y-5)^2 = 50 \checkmark}}$$

6. Functions, f and g , are given by $f(x) = 3 + \cos x$ and $g(x) = 2x$, $x \in \mathbb{R}$.

(a) Find expressions for

(i) $f(g(x))$ and

2

(ii) $g(f(x))$.

1

(b) Determine the value(s) of x for which $f(g(x)) = g(f(x))$ where $0 \leq x < 2\pi$.

6

a) i) $f(g(x)) = f(2x) \checkmark$
 $f(g(x)) = 3 + \cos 2x \checkmark$

ii) $g(f(x)) = g(3 + \cos x)$

$g(f(x)) = 2(3 + \cos x)$

$g(f(x)) = 6 + 2\cos x \checkmark$

b) $f(g(x)) = g(f(x))$

$\therefore 3 + \cos 2x = 6 + 2\cos x \checkmark$

$\cos 2x - 2\cos x - 3 = 0$

\downarrow
 $2\cos^2 x - 1 - 2\cos x - 3 = 0 \checkmark$

$2\cos^2 x - 2\cos x - 4 = 0 \checkmark$

$2(\cos^2 x - \cos x - 2) = 0$

$2(\cos x - 2)(\cos x + 1) = 0 \checkmark$

\downarrow
 $\cos x - 2 = 0$

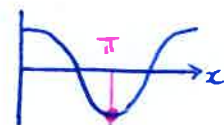
~~$\cos x = 2$~~

No solutions

\downarrow
 $\cos x + 1 = 0$

$\cos x = -1 \checkmark$

$x = \pi$ \checkmark



7. (a) (i) Show that $(x-2)$ is a factor of $2x^3 - 3x^2 - 3x + 2$. 2

(ii) Hence, factorise $2x^3 - 3x^2 - 3x + 2$ fully. 2

The fifth term, u_5 , of a sequence is $u_5 = 2a - 3$.

The terms of the sequence satisfy the recurrence relation $u_{n+1} = au_n - 1$.

(b) Show that $u_7 = 2a^3 - 3a^2 - a - 1$. 1

For this sequence, it is known that

- $u_7 = u_5$
- a limit exists.

(c) (i) Determine the value of a . 3

(ii) Calculate the limit. 1

a) i) Check factor $(x-2)$

$$\begin{array}{r|rrrr} 2 & 2 & -3 & -3 & 2 \\ & \downarrow & & & \\ \hline & 2 & 1 & -1 & 0 \end{array}$$

Since remainder is zero, then $x=2$ is a root and $(x-2)$ is a factor! ✓

ii) $2x^3 - 3x^2 - 3x + 2$

$= (x-2)(2x^2 + x - 1)$ ✓

$= (x-2)(2x-1)(x+1)$ ✓

$$1) b) \quad U_{n+1} = aU_n - 1 \quad U_5 = 2a - 3$$

$$U_6 = a(2a - 3) - 1$$

$$U_6 = 2a^2 - 3a - 1$$

$$U_7 = aU_6 - 1$$

$$U_7 = a(2a^2 - 3a - 1) - 1$$

$$\underline{U_7 = 2a^3 - 3a^2 - a - 1} \quad \text{as required.}$$

$$c) i) \quad \text{If } U_7 = U_5$$

$$\text{then } 2a^3 - 3a^2 - a - 1 = 2a - 3$$

$$2a^3 - 3a^2 - 3a + 2 = 0 \quad \checkmark$$

$$(a-2)(2a-1)(a+1) = 0$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ a-2=0 & 2a-1=0 & a+1=0 \\ a=2 \text{ (or)} & a=\frac{1}{2} & \text{(or)} \quad a=-1 \quad \checkmark \end{array}$$

For a limit to exist, $-1 < a < 1$, then $\underline{a = \frac{1}{2}}$ \checkmark

$$ii) \quad \text{Limit, } L = \frac{-1}{1 - \frac{1}{2}}$$

$$L = \frac{-1}{\frac{1}{2}}$$

$$\underline{\underline{\text{Limit, } L = -2}} \quad \checkmark$$

8. (a) Express $2\cos x^\circ - \sin x^\circ$ in the form $k\cos(x-a)^\circ$, $k > 0$, $0 < a < 360$.

4

(b) Hence, or otherwise, find

(i) the minimum value of $6\cos x^\circ - 3\sin x^\circ$ and

1

(ii) the value of x for which it occurs where $0 \leq x < 360$.

2

$$\begin{aligned} \text{a) } k\cos(x-a) &= k\cos x \cos a + k\sin x \sin a \checkmark \\ &= k\cos a \cdot \cos x + k\sin a \cdot \sin x \\ &\rightarrow 2 \cdot \cos x - 1 \cdot \sin x \end{aligned}$$

$$k\cos a = 2$$

$$k\sin a = -1 \checkmark$$

$$k = \sqrt{2^2 + (-1)^2}$$

$$k = \sqrt{5} \checkmark$$

$$\tan a = \frac{k\sin a}{k\cos a}$$

$$\tan a = \frac{-1}{2}$$



$$a = \tan^{-1}\left(\frac{1}{2}\right) = 26.6^\circ$$

$$\text{Quad. 4} \therefore a = 360 - 26.6 = 333.4^\circ$$

$$\hookrightarrow \underline{2\cos x^\circ - \sin x^\circ = \sqrt{5}\cos(x - 333.4)^\circ} \checkmark$$

b) Consider $y = 6\cos x - 3\sin x$

$$y = 3(2\cos x - \sin x)$$

$$y = 3\sqrt{5}\cos(x - 333.4)^\circ$$

↑
Amplitude $3\sqrt{5}$

↑
shift right by 333.4°



i) Minimum value = $-3\sqrt{5}$ \checkmark

ii) $x = 180 + 333.4$

$$x = 513.4^\circ \checkmark$$

$$x = 153.4^\circ \checkmark$$

9. A sector with a particular fixed area has radius x cm.

The perimeter, P cm, of the sector is given by

$$P = 2x + \frac{128}{x}$$

Find the minimum value of P .

6

$$P = 2x + 128x^{-1} \quad \checkmark$$

$$P' = 2 - 128x^{-2} \quad \checkmark$$

$$P' = 2 - \frac{128}{x^2}$$

Stationary Points occur at $P' = 0$

$$0 = 2 - \frac{128}{x^2} \quad \checkmark$$

$$\frac{128}{x^2} = 2$$

$$128 = 2x^2$$

$$64 = x^2$$

$$x = \pm 8 \quad \text{but } x > 0 \text{ for radius}$$

$$\underline{x = 8 \text{ cm.}} \quad \checkmark$$

Nature Table

x	$\xrightarrow{7}$	8	$\xrightarrow{9}$
P'	-	0	+
slope	\	-	/

$$P'(7) = 2 - \frac{128}{49} = -\text{ve}$$

$$P'(9) = 2 - \frac{128}{81} = +\text{ve}$$

The Perimeter minimises
when radius, $x = 8$ cm.

At $x = 8$,

$$P = 2(8) + \frac{128}{8}$$

$$\underline{\underline{P = 32 \text{ cm}}} \quad \checkmark$$

10. The equation $x^2 + (m-3)x + m = 0$ has two real and distinct roots.

Determine the range of values for m .

4

$$x^2 + (m-3)x + m = 0$$
$$ax^2 + bx + c = 0$$

$$a = 1$$

$$b = (m-3)$$

$$c = m$$

$b^2 - 4ac > 0$ for real + distinct roots

$$(m-3)^2 - 4(1)(m) > 0 \quad \checkmark \checkmark$$

$$m^2 - 6m + 9 - 4m > 0$$

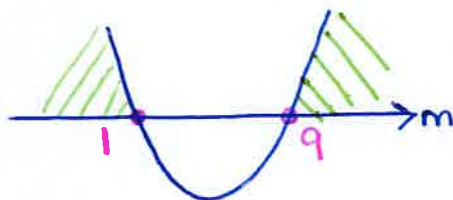
$$* \quad m^2 - 10m + 9 > 0 \quad *$$

consider roots at $m^2 - 10m + 9 = 0$

$$(m-9)(m-1) = 0$$

$$\downarrow \quad \downarrow$$
$$m=9 \quad \text{and} \quad m=1 \quad \checkmark$$

Sketch:



Solutions:

For real and distinct roots,

$$\underline{\underline{m < 1 \quad \text{or} \quad m > 9}} \quad \checkmark$$

11. A supermarket has been investigating how long customers have to wait at the checkout. During any half hour period, the percentage, $P\%$, of customers who wait for less than t minutes, can be modelled by

$$P = 100(1 - e^{kt}), \text{ where } k \text{ is a constant.}$$

- (a) If 50% of customers wait for less than 3 minutes, determine the value of k . 4
- (b) Calculate the percentage of customers who wait for 5 minutes or longer. 2

a) Let $P = 50$ at $t = 3$

$$50 = 100(1 - e^{3k}) \checkmark$$

$$\frac{1}{2} = 1 - e^{3k}$$

$$e^{3k} = \frac{1}{2} \checkmark$$

$$\ln e^{3k} = \ln \frac{1}{2}$$

$$3k = \ln \frac{1}{2} \checkmark$$

$$k = \frac{\ln \frac{1}{2}}{3}$$

$$\underline{k = -0.231} \checkmark$$

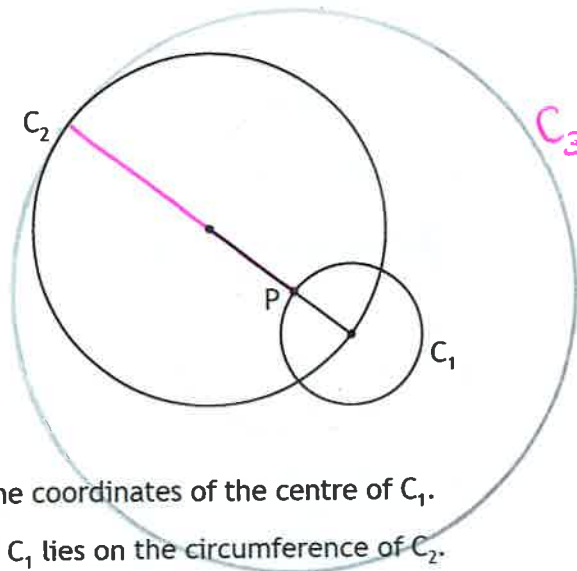
b) At $t = 5$

$$P = 100(1 - e^{-0.231(5)})$$

$$P = 68.4942... \checkmark$$

As 68.5% of customers wait less than 5 mins,
then 31.5% of customers wait longer. \checkmark

12. Circle C_1 has equation $(x-13)^2 + (y+4)^2 = 100$.
 Circle C_2 has equation $x^2 + y^2 + 14x - 22y + c = 0$.



- (a) (i) Write down the coordinates of the centre of C_1 . 1
 (ii) The centre of C_1 lies on the circumference of C_2 .
 Show that $c = -455$. 1

The line joining the centres of the circles intersects C_1 at P.

- (b) (i) Determine the ratio in which P divides the line joining the centres of the circles. 2
 (ii) Hence, or otherwise, determine the coordinates of P. 2

P is the centre of a third circle, C_3 .

C_2 touches C_3 internally.

- (c) Determine the equation of C_3 . 1

a) i) $C_1: (x-13)^2 + (y+4)^2 = 100$
 $(x-a)^2 + (y-b)^2 = r^2$

Centre $C_1 (13, -4)$ ✓

ii) $C_2: x^2 + y^2 + 14x - 22y + c = 0$
 $(13)^2 + (-4)^2 + 14(13) - 22(-4) + c = 0$
 $169 + 16 + 182 + 88 + c = 0$
 $455 + c = 0$

$c = -455$ ✓

as required.

12) b) Consider radius of each circle

$$r_1 = \sqrt{100}$$

$$r_1 = 10$$

$$C_2: x^2 + y^2 + 14x - 22y - 455 = 0$$

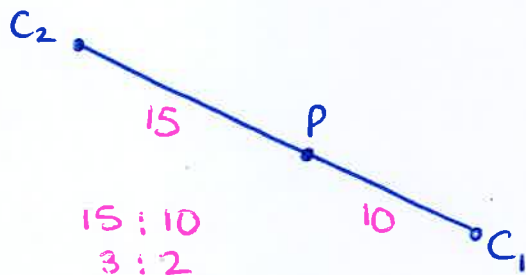
$$2g = 14 \quad 2f = -22 \quad c = -455$$

$$g = 7 \quad f = -11$$

$$r_2 = \sqrt{7^2 + (-11)^2 - (-455)}$$

$$r_2 = \sqrt{625}$$

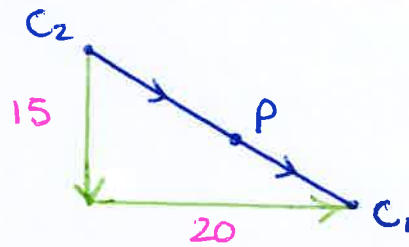
$$r_2 = 25 \quad \checkmark$$



i) P divides the line in the ratio 3:2 ✓

ii) $C_1 (13, -4)$

$C_2 (-7, 11)$ ✓



$$\vec{C_2 C_1} = \begin{pmatrix} 20 \\ -15 \end{pmatrix} \quad \therefore \quad \vec{C_2 P} = \frac{3}{5} \begin{pmatrix} 20 \\ -15 \end{pmatrix} = \begin{pmatrix} 12 \\ -9 \end{pmatrix}$$

$$P = \begin{pmatrix} -7 \\ 11 \end{pmatrix} + \begin{pmatrix} 12 \\ -9 \end{pmatrix}$$

$$P = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad \therefore \quad \text{Point } P(5, 2) \quad \checkmark$$

c) Circle C_3 :

Centre $P(5, 2)$

$$r_3 = r_2 + 15$$

$$r_3 = 25 + 15$$

$$r_3 = 40$$

Equation of C_3

$$(x - a)^2 + (y - b)^2 = r^2$$

$$(x - 5)^2 + (y - 2)^2 = 40^2$$

$$(x - 5)^2 + (y - 2)^2 = 1600 \quad \checkmark$$

