H

National Qualifications 2017

X747/76/12

Mathematics Paper 2

FRIDAY, 5 MAY

10:30 AM – 12:00 NOON

70 marks

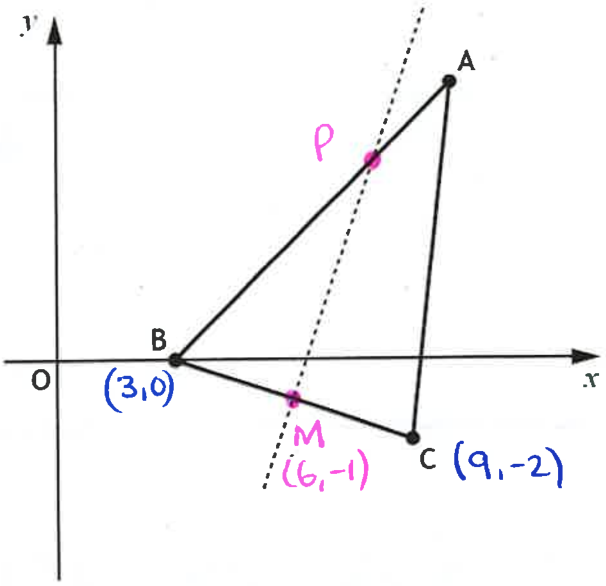
2017 PAPER 2 – WORKED SOLUTIONS

H Wallace

1. Triangle ABC is shown in the diagram below.

The coordinates of B are (3,0) and the coordinates of C are (9,–2).

The broken line is the perpendicular bisector of BC.



(a) Find the equation of the perpendicular bisector of BC. **4**

(b) The line AB makes an angle of 45° with the positive direction of the x-axis.

Find the equation of AB. **2**

(c) Find the coordinates of the point of intersection of AB and the perpendicular bisector of BC. **2**

a) Midpoint of BC

$M\left(\frac{3+9}{2}, \frac{0+-2}{2}\right)$

$M(6,-1)$ ✓

Gradient of BC

$m_{BC} = \frac{-2-0}{9-3} = \frac{-2}{6} = -\frac{1}{3}$ ✓

Perpendicular gradient

$m_{\perp} = 3$ ✓ since $m_1 \times m_2 = -1$

Equation of ⊥ bisector

$y - b = m(x-a)$

$y + 1 = 3(x-6)$

$y + 1 = 3x - 18$

$y = 3x - 19$ ✓

b) $m_{AB} = \tan 45°$

$m_{AB} = 1$ ✓

Equation of AB

$y - b = m(x - a)$

$y - 0 = 1(x - 3)$

$y = x - 3$ ✓

c) Point of Intersection

Let $y = y$

$3x - 19 = x - 3$

$2x = 16$

$x = 8$ ✓

$y = 8 - 3$

$y = 5$

Point of Intersection $(8, 5)$ ✓

2. (a) Show that $(x-1)$ is a factor of $f(x) = 2x^3 - 5x^2 + x + 2$. 2

(b) Hence, or otherwise, solve $f(x) = 0$. 3

a)

1	2	−5	1	2
	↓	2	−3	−2
	2	−3	−2	$\boxed{0}$

✓

Since the remainder is zero, then $x = 1$ is a root and $(x-1)$ is a factor of $f(x)$. ✓

b) $f(x) = 0$

$2x^3 - 5x^2 + x + 2 = 0$

$(x-1)(2x^2 - 3x - 2) = 0$ ✓

$(x-1)(2x+1)(x-2) = 0$ ✓

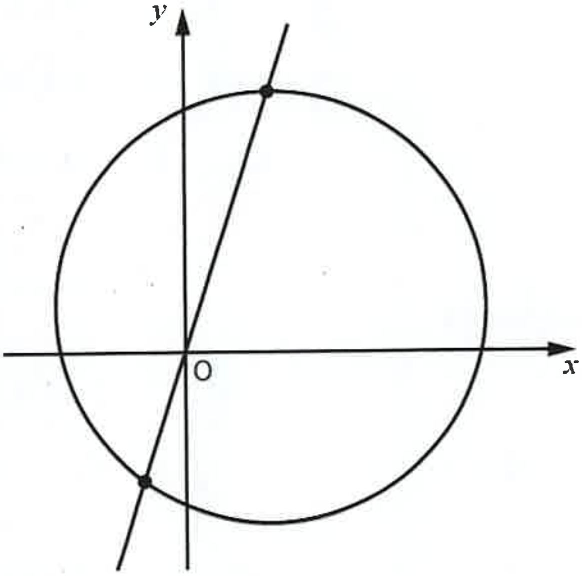
$x - 1 = 0$, $x = 1$

$2x + 1 = 0$, $x = -\frac{1}{2}$

$x - 2 = 0$, $x = 2$

↳ $x = -\frac{1}{2}, 1, 2$ ✓

3. The line $y = 3x$ intersects the circle with equation $(x-2)^2 + (y-1)^2 = 25$.



Find the coordinates of the points of intersection. **5**

Let $y = 3x$ in circle

$(x-2)^2 + (3x-1)^2 = 25$

$x^2 - 4x + 4 + 9x^2 - 6x + 1 = 25$

$10x^2 - 10x - 20 = 0$

$10(x^2 - x - 2) = 0$

$10(x-2)(x+1) = 0$

$x - 2 = 0$ $\quad$ $x + 1 = 0$

$x = 2$ $\quad$ $x = -1$

$y = 3(2)$ $\quad$ $y = 3(-1)$

$y = 6$ $\quad$ $y = -3$

Points $(2, 6)$ and $(-1, -3)$

4. (a) Express $3x^2+24x+50$ in the form $a(x+b)^2+c$. 3

(b) Given that $f(x)=x^3+12x^2+50x-11$, find $f'(x)$. 2

(c) Hence, or otherwise, explain why the curve with equation $y=f(x)$ is strictly increasing for all values of x. 2

a) $[3x^2+24x]+50$

$3[x^2+8x]+50$ ✓

$3[(x+4)^2-16]+50$ ✓

$3(x+4)^2-48+50$

$3(x+4)^2+2$ ✓

b) $f(x) = x^3+12x^2+50x-11$

$f'(x) = 3x^2+24x+50$ ✓✓

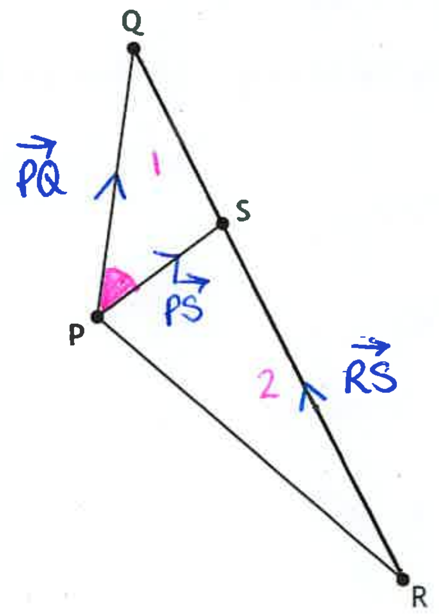
c) ↳ $f'(x) = 3(x+4)^2+2$

Since $(x+4)^2 \geqslant 0$ for all values of x, ✓

then $3(x+4)^2+2 > 0$.

$\therefore$ $f'(x) > 0$ for all values of x and $y=f(x)$ is strictly increasing. ✓

5. In the diagram, $\overrightarrow{PR} = 9\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{RQ} = -12\mathbf{i} - 9\mathbf{j} + 3\mathbf{k}$.



(a) Express $\overrightarrow{PQ}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. **2**

The point S divides QR in the ratio 1:2.

(b) Show that $\overrightarrow{PS} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$. **2**

(c) Hence, find the size of angle QPS. **5**

a) $\overrightarrow{PQ} = \overrightarrow{PR} + \overrightarrow{RQ}$ ✓

$= \begin{pmatrix} 9 \\ 5 \\ 2 \end{pmatrix} + \begin{pmatrix} -12 \\ -9 \\ 3 \end{pmatrix}$

$\overrightarrow{PQ} = \begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix}$

$\overrightarrow{PQ} = -3i - 4j + 5k$ ✓

5) b) $\overrightarrow{PS} = \overrightarrow{PR} + \overrightarrow{RS}$

$\overrightarrow{PS} = \overrightarrow{PR} + \frac{2}{3}\overrightarrow{RQ}$

$\overrightarrow{PS} = \begin{pmatrix} 9 \\ 5 \\ 2 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} -12 \\ -9 \\ 3 \end{pmatrix}$

$\overrightarrow{PS} = \begin{pmatrix} 9 \\ 5 \\ 2 \end{pmatrix} + \begin{pmatrix} -8 \\ -6 \\ 2 \end{pmatrix}$

$\overrightarrow{PS} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$

$\overrightarrow{PS} = i - j + 4k$ as required.

c) $\cos QPS = \dfrac{\overrightarrow{PQ} \cdot \overrightarrow{PS}}{|\overrightarrow{PQ}|\,|\overrightarrow{PS}|}$

$|\overrightarrow{PQ}| = \sqrt{(-3)^2 + (-4)^2 + (5)^2}$

$|\overrightarrow{PQ}| = \sqrt{50}$

$|\overrightarrow{PS}| = \sqrt{(1)^2 + (-1)^2 + (4)^2}$

$|\overrightarrow{PS}| = \sqrt{18}$

$\overrightarrow{PQ} \cdot \overrightarrow{PS} = (-3)(1) + (-4)(-1) + (5)(4)$

$\overrightarrow{PQ} \cdot \overrightarrow{PS} = -3 + 4 + 20$

$\overrightarrow{PQ} \cdot \overrightarrow{PS} = 21$

$\cos QPS = \dfrac{21}{\sqrt{50}\sqrt{18}}$

$\cos QPS = \dfrac{21}{30}$

$QPS = \cos^{-1}(21/30)$

$QPS = 45.572996$

Angle $QPS = 45.6^\circ$ (1dp)

6. Solve $5\sin x - 4 = 2\cos 2x$ for $0 \le x < 2\pi$. **5**

$$5\sin x - 4 = 2\cos 2x$$

$$5\sin x - 4 = 2(1 - 2\sin^2 x)$$

$$5\sin x - 4 = 2 - 4\sin^2 x$$

$$4\sin^2 x + 5\sin x - 6 = 0$$

Consider $4s^2 + 5s - 6 = 0$

$$(4s-3)(s+2) = 0$$

$$\hookrightarrow (4\sin x - 3)(\sin x + 2) = 0$$

$4\sin x - 3 = 0$

S	A
T	C

$\sin x = \frac{3}{4}$

$\sin x + 2 = 0$

$\sin x = -2$

No solution

$ra = \sin^{-1}(3/4) = 0.848\ldots$

$$x = 0.848,\ \pi - 0.848$$

$$x = 0.848,\ 2.294$$

7. (a) Find the x-coordinate of the stationary point on the curve with equation $y = 6x - 2\sqrt{x^3}$. 4

(b) Hence, determine the greatest and least values of y in the interval $1 \le x \le 9$. 3

a) $y = 6x - 2\sqrt{x^3}$

$y = 6x - 2x^{3/2}$ ✓

$\frac{dy}{dx} = 6$ ✓ $- 3x^{1/2}$

Stationary Points occur at $\frac{dy}{dx} = 0$

$6 - 3x^{1/2} = 0$ ✓

$6 - 3\sqrt{x} = 0$

$6 = 3\sqrt{x}$

$2 = \sqrt{x}$

$x = 4$ ✓

b) Consider y-values at $x = 1, 4$ and 9.

At $x = 1$

$y = 6(1) - 2\sqrt{1^3}$

$y = 6 - 2\sqrt{1}$

$y = 6 - 2$

$y = 4$

Point $(1, 4)$

At $x = 4$

$y = 6(4) - 2\sqrt{4^3}$

$y = 24 - 2(8)$

$y = 24 - 16$

$y = 8$ ✓

Point $(4, 8)$
max TP.

At $x = 9$

$y = 6(9) - 2\sqrt{9^3}$

$y = 54 - 2(27)$

$y = 54 - 54$

$y = 0$ ✓

Point $(9, 0)$

In the interval $1 \leq x \leq 9$,

greatest value is 8, and least value is 0. ✓

8. Sequences may be generated by recurrence relations of the form $u_{n+1} = ku_n - 20$, $u_0 = 5$ where $k \in \mathbb{R}$.

(a) Show that $u_2 = 5k^2 - 20k - 20$. **2**

(b) Determine the range of values of k for which $u_2 < u_0$. **4**

a) $U_1 = kU_0 - 20$

$U_1 = 5k - 20$ ✓

$U_2 = kU_1 - 20$

$U_2 = k(5k-20) - 20$

$U_2 = 5k^2 - 20k - 20$ ✓

b) For $U_2 < U_0$.

then $5k^2 - 20k - 20 < 5$ ✓

* $5k^2 - 20k - 25 < 0$ * ✓

consider roots of $5k^2 - 20k - 25 = 0$

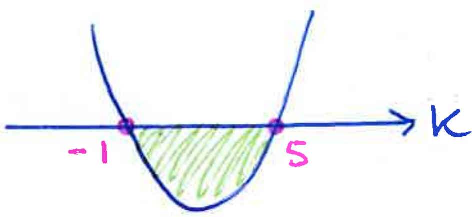
$5(k^2 - 4k - 5) = 0$

$5(k-5)(k+1) = 0$

$k - 5 = 0$ $\quad$ $k + 1 = 0$

$k = 5$ $\quad$ $k = -1$ ✓

Sketch of parabola:

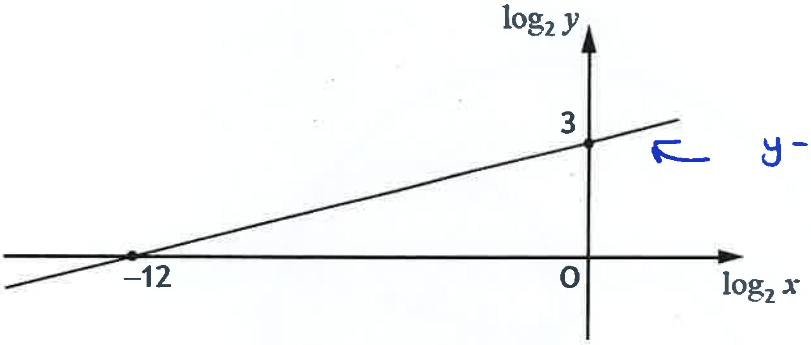


For $U_2 < U_0$

then $-1 < k < 5$ ✓

9. Two variables, x and y, are connected by the equation $y = kx^n$.

The graph of $\log_2 y$ against $\log_2 x$ is a straight line as shown.



y-intercept, c=3

Find the values of k and n. **5**

Consider gradient between $(-12, 0)$ and $(0,3)$

$$m = \frac{3-0}{0-(-12)} = \frac{3}{12} = \frac{1}{4}$$

If $Y = mX + c$

then $\log_2 y = \frac{1}{4}\log_2 x + 3$ ✓

$\log_2 y = \frac{1}{4}\log_2 x + 3\log_2 2$ ✓

$\log_2 y = \log_2 x^{1/4} + \log_2 2^3$ ✓

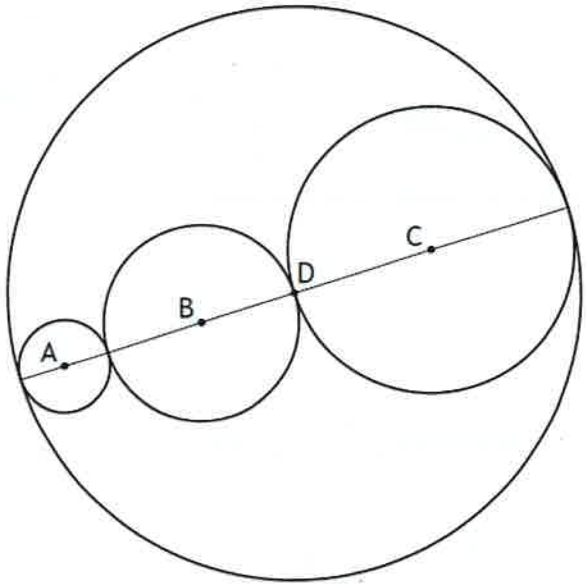
$\log_2 y = \log_2 8x^{1/4}$ ✓

$y = 8x^{1/4}$ ✓

In the form $y = kx^n$, $k = 8$ and $n = \frac{1}{4}$

10. (a) Show that the points A(–7, –2), B(2, 1) and C(17, 6) are collinear. **3**

Three circles with centres A, B and C are drawn inside a circle with centre D as shown.

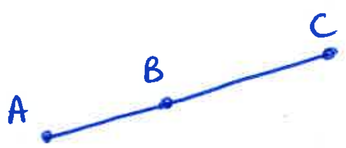


The circles with centres A, B and C have radii r_A, r_B and r_C respectively.

- $r_A = \sqrt{10}$
- $r_B = 2r_A$
- $r_C = r_A + r_B$

(b) Determine the equation of the circle with centre D. **4**

a)



$m_{AB} = \dfrac{1-(-2)}{2-(-7)} = \dfrac{3}{9} = \dfrac{1}{3}$ ✓ $m_{BC} = \dfrac{6-1}{17-2} = \dfrac{5}{15} = \dfrac{1}{3}$ ✓

Since $m_{AB} = m_{BC}$, then line AB is parallel to BC. As B is a common point, then points A, B and C are collinear. ✓

10) b) Radius, $r_D = 2r_C$

$r_A = \sqrt{10}$

$r_D = 2(r_A + r_B)$

$r_B = 2r_A$

$r_D = 2(\sqrt{10} + 2\sqrt{10})$

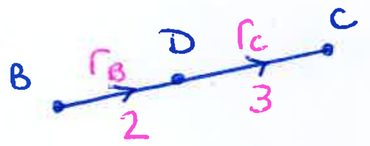
$r_B = 2\sqrt{10}$

$r_D = 2(3\sqrt{10})$

$r_C = 3\sqrt{10}$

$r_D = 6\sqrt{10}$ ✓

Centre D



D divides BC in the ratio 2 : 3.

$$\frac{\overrightarrow{BD}}{\overrightarrow{DC}} = \frac{2}{3}$$ ✓

$$3\overrightarrow{BD} = 2\overrightarrow{DC}$$

$$3(d - b) = 2(c - d)$$

$$3d - 3b = 2c - 2d$$

$$5d = 2c + 3b$$

$$5d = 2\begin{pmatrix}17\\6\end{pmatrix} + 3\begin{pmatrix}2\\1\end{pmatrix}$$

$$5d = \begin{pmatrix}34\\12\end{pmatrix} + \begin{pmatrix}6\\3\end{pmatrix}$$

$$5d = \begin{pmatrix}40\\15\end{pmatrix}$$

$$d = \begin{pmatrix}8\\3\end{pmatrix}$$

Centre D (8, 3) ✓

Equation of Circle

$$(x-a)^2 + (y-b)^2 = r^2$$

$$(x-8)^2 + (y-3)^2 = (6\sqrt{10})^2$$

$$(x-8)^2 + (y-3)^2 = 360.$$ ✓

11. (a) Show that $\dfrac{\sin 2x}{2\cos x} - \sin x \cos^2 x = \sin^3 x$, where $0 < x < \dfrac{\pi}{2}$. **3**

(b) Hence, differentiate $\dfrac{\sin 2x}{2\cos x} - \sin x \cos^2 x$, where $0 < x < \dfrac{\pi}{2}$. **3**

a) $LHS = \dfrac{\sin 2x}{2\cos x} - \sin x \cos^2 x$

$LHS = \dfrac{2\sin x \cos x}{2\cos x} - \sin x \cos^2 x$ ✓

$LHS = \sin x - \sin x \cos^2 x$

$LHS = \sin x\,(1 - \cos^2 x)$ ✓

$LHS = \sin x\,(\sin^2 x)$

$LHS = \sin^3 x$ ✓

$LHS = RHS$ as required.

b) $\dfrac{d}{dx}\left(\dfrac{\sin 2x}{2\cos x} - \sin x \cos^2 x\right)$

$= \dfrac{d}{dx}(\sin^3 x)$ ✓

$= \dfrac{d}{dx}(\sin x)^3$

$= 3(\sin x)^2 \times \cos x$ ✓

$= 3\sin^2 x \cos x$ ✓