



National
Qualifications
2017

X747/76/11

**Mathematics
Paper 1
(Non-Calculator)**

FRIDAY, 5 MAY

9:00 AM – 10:10 AM

60 marks

2017 PAPER 1 – WORKED SOLUTIONS

H Wallace

1. Functions f and g are defined on suitable domains by $f(x) = 5x$ and $g(x) = 2\cos x$.

(a) Evaluate $f(g(0))$.

1

(b) Find an expression for $g(f(x))$.

2

$$\begin{aligned} \text{a) } g(0) &= 2\cos 0 \\ g(0) &= 2(1) \\ g(0) &= 2 \end{aligned}$$

$$\begin{aligned} f(g(0)) &= f(2) \\ &= 5(2) \end{aligned}$$

$$\underline{\underline{f(g(0)) = 10 \quad \checkmark}}$$

$$\text{b) } g(f(x)) = g(5x) \quad \checkmark$$

$$\underline{\underline{g(f(x)) = 2\cos 5x \quad \checkmark}}$$

2. The point $P(-2, 1)$ lies on the circle $x^2 + y^2 - 8x - 6y - 15 = 0$.

Find the equation of the tangent to the circle at P .

4

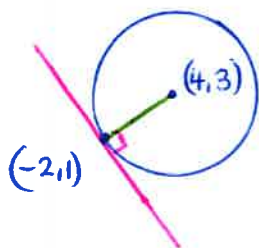
$$x^2 + y^2 - 8x - 6y - 15 = 0$$

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$\begin{aligned} 2g &= -8 \\ g &= -4 \end{aligned}$$

$$\begin{aligned} 2f &= -6 \\ f &= -3 \end{aligned}$$

Centre $(4, 3) \quad \checkmark$



Gradient Radius

$$m_{\text{rad}} = \frac{3-1}{4-(-2)} = \frac{2}{6} = \frac{1}{3} \quad \checkmark$$

Gradient Tangent

$$m_{\text{tan}} = -3 \quad \checkmark \quad \text{since } m_{\text{rad}} \times m_{\text{tan}} = -1$$

Equation of tangent

$$m = -3$$

$$a = -2$$

$$b = 1$$

$$y - b = m(x - a)$$

$$y - 1 = -3(x + 2)$$

$$y - 1 = -3x - 6$$

$$\underline{\underline{y = -3x - 5 \quad \checkmark}}$$

3. Given $y = (4x-1)^{12}$, find $\frac{dy}{dx}$.

2

$$\frac{dy}{dx} = 12(4x-1)^{11} \times 4$$

$$\frac{dy}{dx} = 48(4x-1)^{11}$$

4. Find the value of k for which the equation $x^2 + 4x + (k-5) = 0$ has equal roots.

3

$$ax^2 + bx + c = 0$$

$$a = 1$$

$$b = 4$$

$$c = k-5$$

$$b^2 - 4ac = 0 \text{ for equal roots}$$

$$(4)^2 - 4(1)(k-5) = 0$$

$$16 - 4k + 20 = 0$$

$$36 - 4k = 0$$

$$36 = 4k$$

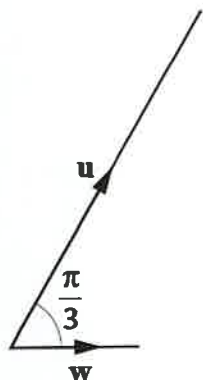
$$k = 9$$

5. Vectors u and v are $\begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -8 \\ 6 \end{pmatrix}$ respectively.

(a) Evaluate $u \cdot v$.

1

(b)



Vector w makes an angle of $\frac{\pi}{3}$ with u and $|w| = \sqrt{3}$.

Calculate $u \cdot w$.

3

$$\begin{aligned} \text{a) } u \cdot v &= 5(3) + 1(-8) + (-1)(6) \\ u \cdot v &= 15 - 8 - 6 \\ u \cdot v &= 1. \checkmark \end{aligned}$$

$$\text{b) } u \cdot w = |u| |w| \cos \frac{\pi}{3}$$

$$u \cdot w = (3\sqrt{3})(\sqrt{3}) \cos \frac{\pi}{3} \checkmark$$

$$u \cdot w = 9 \cos \frac{\pi}{3}$$

$$u \cdot w = 9 \left(\frac{1}{2} \right)$$

$$u \cdot w = \frac{9}{2}$$

$$u \cdot w = 4\frac{1}{2} \checkmark$$

$$|u| = \sqrt{5^2 + 1^2 + (-1)^2}$$

$$|u| = \sqrt{25 + 1 + 1}$$

$$|u| = \sqrt{27} \checkmark$$

$$|u| = 3\sqrt{3}$$

$$\cos \frac{\pi}{3} = \cos 60^\circ$$

$$= \frac{1}{2}$$



6. A function, h , is defined by $h(x) = x^3 + 7$, where $x \in \mathbb{R}$.

Determine an expression for $h^{-1}(x)$.

3

$$\text{Let } y = x^3 + 7$$

$$y - 7 = x^3 \quad \checkmark$$

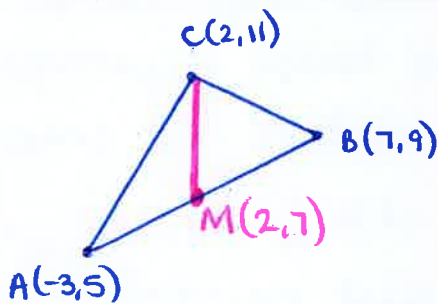
$$x = \sqrt[3]{y - 7} \quad \checkmark$$

$$\hookrightarrow \underline{\underline{h^{-1}(x) = \sqrt[3]{x - 7}}} \quad \checkmark$$

7. $A(-3, 5)$, $B(7, 9)$ and $C(2, 11)$ are the vertices of a triangle.

Find the equation of the median through C .

3



Midpoint of AB .

$$M \left(\frac{-3+7}{2}, \frac{5+9}{2} \right)$$

$$\underline{\underline{M(2, 7)}} \quad \checkmark$$

Gradient of Median

$$m_{\text{med}} = \frac{11-7}{2-2} = \frac{4}{0}$$

Gradient is undefined! \checkmark

\therefore Median is vertical

Equation of Median

$$\underline{\underline{x = 2}} \quad \checkmark$$

8. Calculate the rate of change of $d(t) = \frac{1}{2t}$, $t \neq 0$, when $t = 5$.

3

$$d(t) = \frac{1}{2} t^{-1} \checkmark$$

$$d'(t) = -\frac{1}{2} t^{-2} \checkmark$$

$$d'(t) = -\frac{1}{2t^2}$$

$$d'(5) = -\frac{1}{2(5)^2}$$

$$d'(5) = -\frac{1}{50} \checkmark$$

9. A sequence is generated by the recurrence relation $u_{n+1} = mu_n + 6$ where m is a constant.

(a) Given $u_1 = 28$ and $u_2 = 13$, find the value of m .

2

(b) (i) Explain why this sequence approaches a limit as $n \rightarrow \infty$.

1

(ii) Calculate this limit.

2

a)

$$u_2 = mu_1 + 6$$
$$13 = 28m + 6 \checkmark$$
$$7 = 28m$$
$$m = \frac{1}{4} \checkmark$$

b) i) This sequence is a linear recurrence relation and since $-1 < \frac{1}{4} < 1$, a limit exists.

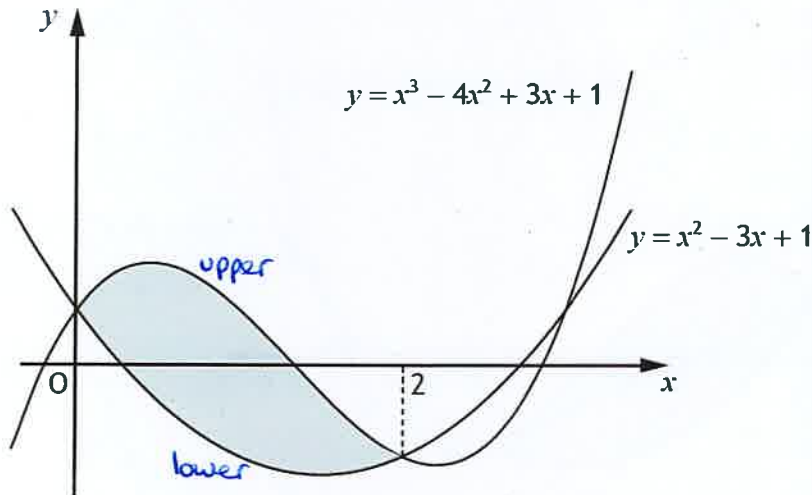
b) ii)

$$u_{n+1} = \frac{1}{4} u_n + 6$$

$$L = \frac{6}{1 - \frac{1}{4}} = \frac{6}{\frac{3}{4}} = \frac{24}{3} = 8$$

$$\text{Limit} = 8 \checkmark$$

10. Two curves with equations $y = x^3 - 4x^2 + 3x + 1$ and $y = x^2 - 3x + 1$ intersect as shown in the diagram.

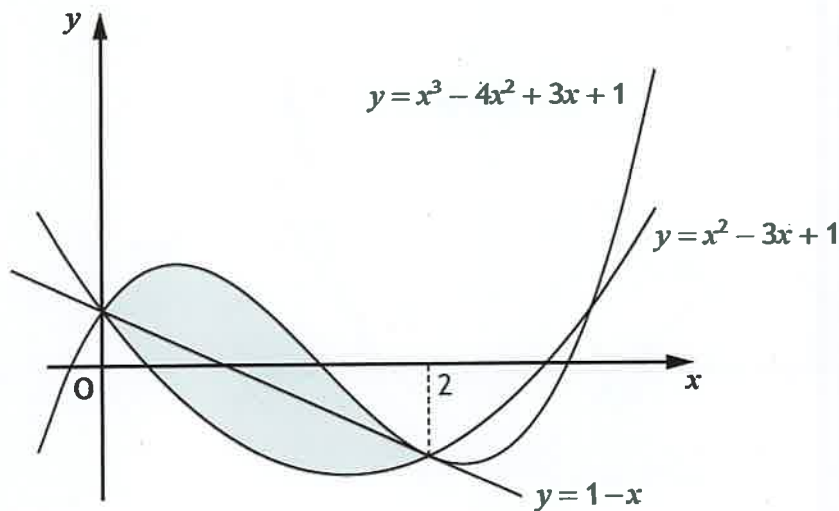


- (a) Calculate the shaded area.

5

$$\begin{aligned}
 \text{Area} &= \int_0^2 (x^3 - 4x^2 + 3x + 1) - (x^2 - 3x + 1) \, dx \\
 &= \int_0^2 x^3 - 4x^2 + 3x + 1 - x^2 + 3x - 1 \, dx \\
 &= \int_0^2 x^3 - 5x^2 + 6x \, dx \\
 &= \left[\frac{x^4}{4} - \frac{5x^3}{3} + \frac{6x^2}{2} \right]_0^2 \\
 &= \left[\frac{x^4}{4} - \frac{5x^3}{3} + 3x^2 \right]_0^2 \\
 &= \left(\frac{(2)^4}{4} - \frac{5(2)^3}{3} + 3(2)^2 \right) - \left(\frac{(0)^4}{4} - \frac{5(0)^3}{3} + 3(0)^2 \right) \\
 &= \left(\frac{16}{4} - \frac{40}{3} + 12 \right) - (0) \\
 &= 4 - 13\frac{1}{3} + 12 \\
 &= 16 - 13\frac{1}{3} \\
 \text{Area} &= \underline{\underline{2\frac{2}{3} \text{ units}^2}}
 \end{aligned}$$

- 10) The line passing through the points of intersection of the curves has equation $y = 1 - x$.



- (b) Determine the fraction of the shaded area which lies below the line $y = 1 - x$.

4

$$\begin{aligned}
 \text{Area below line} &= \int_0^2 (1-x) - (x^2 - 3x + 1) \, dx \checkmark \\
 &= \int_0^2 1 - x - x^2 + 3x - 1 \, dx \\
 &= \int_0^2 2x - x^2 \, dx \\
 &= \left[\frac{2x^2}{2} - \frac{x^3}{3} \right]_0^2 \\
 &= \left[x^2 - \frac{x^3}{3} \right]_0^2 \checkmark \\
 &= \left((2)^2 - \frac{(2)^3}{3} \right) - \left(0^2 - \frac{0^3}{3} \right) \\
 &= \left(4 - \frac{8}{3} \right) - 0 \\
 &= 4 - 2\frac{2}{3} \\
 \text{Area below line} &= 1\frac{1}{3} \text{ units}^2 \checkmark
 \end{aligned}$$

Area below is $\frac{1}{2}$ of the total shaded area. \checkmark

11. A and B are the points $(-7, 2)$ and $(5, a)$.

AB is parallel to the line with equation $3y - 2x = 4$.

Determine the value of a .

3

Gradient of line

$$3y - 2x = 4$$

$$3y = 2x + 4$$

$$y = \frac{2}{3}x + \frac{4}{3}$$

$$m = \frac{2}{3} \quad \checkmark$$

Parallel lines have
equal gradient

$$\therefore m_{AB} = \frac{2}{3}$$

Setup gradient of AB

$$m_{AB} = \frac{a-2}{5-(-7)} = \frac{a-2}{12} \quad \checkmark$$

$$\text{Let } \frac{a-2}{12} = \frac{2}{3}$$

$$a - 2 = 8$$

$$\underline{\underline{a = 10.}} \quad \checkmark$$

12. Given that $\log_a 36 - \log_a 4 = \frac{1}{2}$, find the value of a .

3

$$\log_a \left(\frac{36}{4} \right) = \frac{1}{2}$$

$$\log_a 9 = \frac{1}{2} \quad \checkmark$$

$$\log_a 9 = \frac{1}{2} \log_a a$$

$$\log_a 9 = \log_a a^{1/2}$$

$$9 = a^{1/2} \quad \checkmark$$

$$\sqrt{a} = 9$$

$$\underline{\underline{a = 81}} \quad \checkmark$$

13. Find $\int \frac{1}{(5-4x)^{\frac{1}{2}}} dx, x < \frac{5}{4}$.

4

Prepare!

$$\int (5-4x)^{-\frac{1}{2}} \cdot dx$$

$$= \frac{(5-4x)^{\frac{1}{2}}}{\frac{1}{2} \times (-4)} + C$$

$$= -\frac{1}{2} (5-4x)^{\frac{1}{2}} + C$$

$$= \underline{\underline{-\frac{1}{2} \sqrt{5-4x} + C}}$$

14. (a) Express $\sqrt{3} \sin x^\circ - \cos x^\circ$ in the form $k \sin(x-a)^\circ$, where $k > 0$ and $0 < a < 360$.

4

$$\begin{aligned} k \sin(x-a) &= k \sin x \cos a - k \cos x \sin a \\ &= k \cos a \cdot \sin x - k \sin a \cdot \cos x \\ &\rightarrow \sqrt{3} \cdot \sin x - 1 \cdot \cos x \end{aligned}$$

$$k \sin a = 1$$

$$k \cos a = \sqrt{3}$$

$$k = \sqrt{1^2 + (\sqrt{3})^2}$$

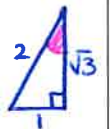
$$k = \sqrt{1+3}$$

$$k = \sqrt{4}$$

$$k = 2$$

$$\tan a = \frac{k \sin a}{k \cos a}$$

$$\tan a = \frac{1}{\sqrt{3}}$$



$$a = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ$$

$$\frac{\text{S}}{\text{T}} \bigg| \frac{\text{A}}{\text{C}} \quad a = 30^\circ$$

$$\rightarrow \underline{\underline{\sqrt{3} \sin x^\circ - \cos x^\circ = 2 \sin(x-30)^\circ}}$$

- (b) Hence, or otherwise, sketch the graph with equation $y = \sqrt{3} \sin x^\circ - \cos x^\circ$, $0 \leq x \leq 360$.

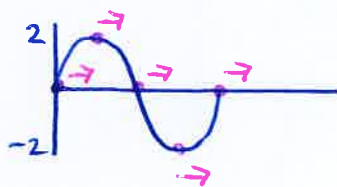
3

Use the diagram provided in the answer booklet.

Sketch $y = 2 \sin(x - 30)^\circ$.

↑
Amplitude 2

↑
shift right by 30°



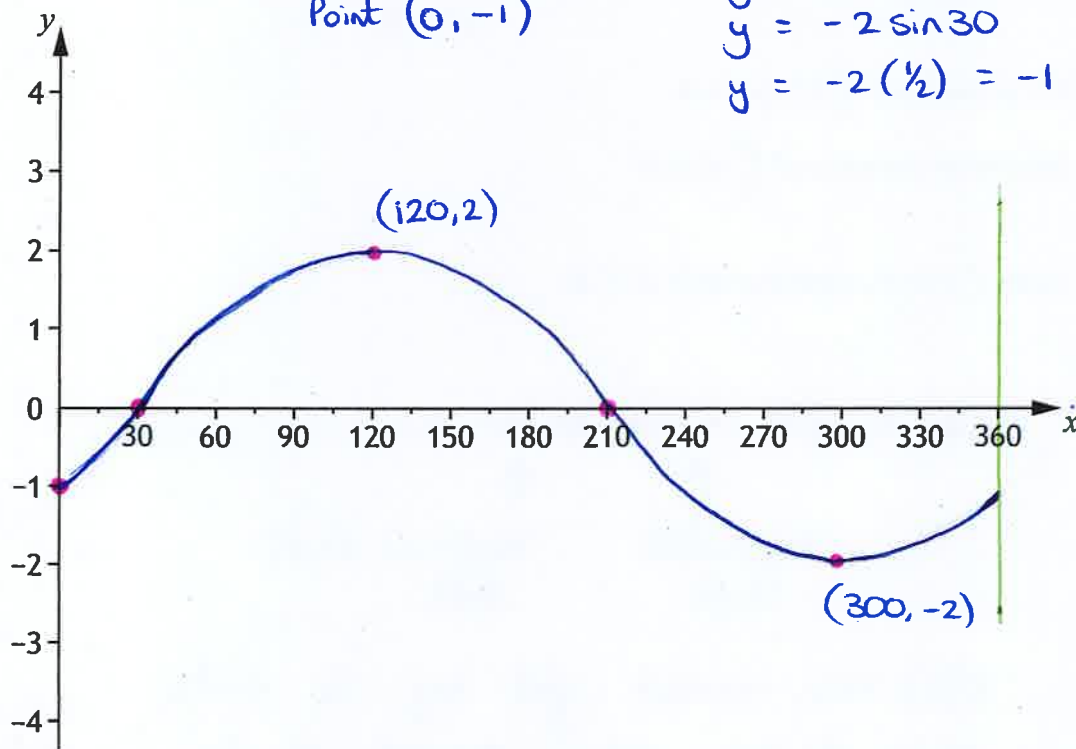
✓ Roots at 30° , 210° , 390°

Max TP at $(120, 2)$

✓ Min TP at $(300, -2)$.

✓ For y-intercept, $x = 0$
Point $(0, -1)$

$$\begin{aligned} y &= 2 \sin(0 - 30) \\ &= 2 \sin(-30) \\ &= -2 \sin 30 \\ &= -2 \left(\frac{1}{2}\right) = -1 \end{aligned}$$



15. A quadratic function, f , is defined on \mathbb{R} , the set of real numbers.

Diagram 1 shows part of the graph with equation $y = f(x)$.

The turning point is $(2, 3)$.

Diagram 2 shows part of the graph with equation $y = h(x)$.

The turning point is $(7, 6)$.

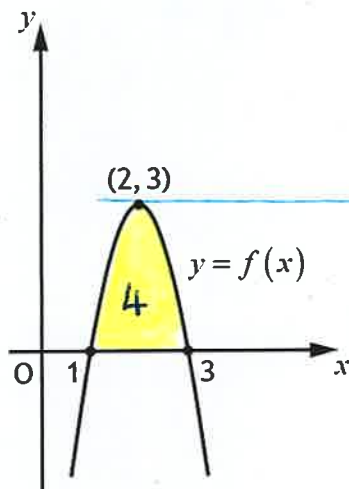


Diagram 1

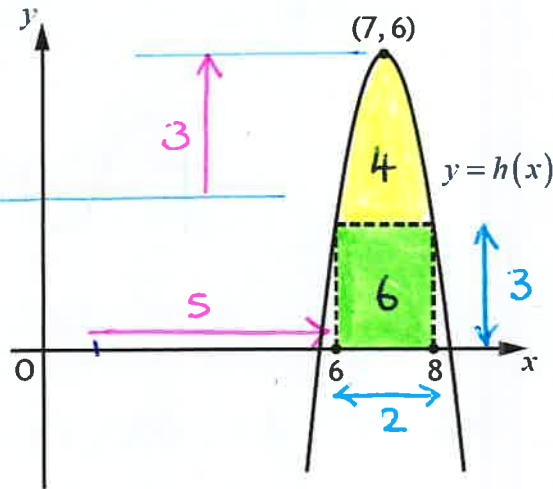


Diagram 2

(a) Given that $h(x) = f(x+a) + b$.

Write down the values of a and b .

2

(b) It is known that $\int_1^3 f(x) dx = 4$.

Determine the value of $\int_6^8 h(x) dx$.

1

(c) Given $f'(1) = 6$, state the value of $h'(8)$.

1

$$a) \quad h(x) = f(x+a) + b$$

↑ horizontal shift ↑ vertical shift.

$f(x)$ has moved right by 5 units.
then it has been moved up by 3 units.

$a = -5$ and $b = 3$ ✓✓

15) b) If $\int_1^3 f(x) \cdot dx = 4$

then $\int_6^8 h(x) dx = 4 + \text{Area rectangle}$

$$= 4 + 2 \times 3$$

$$= 4 + 6$$

$$\hookrightarrow \int_6^8 h(x) \cdot dx = 10 \quad \checkmark$$

c) If $f'(1) = 6$

then $h'(8) = -6 \quad \checkmark$

ie, the gradient of the tangent to $f(x)$ at $x=1$ is 6.

So the gradient of the tangent to $f(x)$ at $x=3$ is -6.

This would be the exact same for the tangent to $h(x)$ at $x=8$.
