H

National
Qualifications
2016

X747/76/12

Mathematics
Paper 2

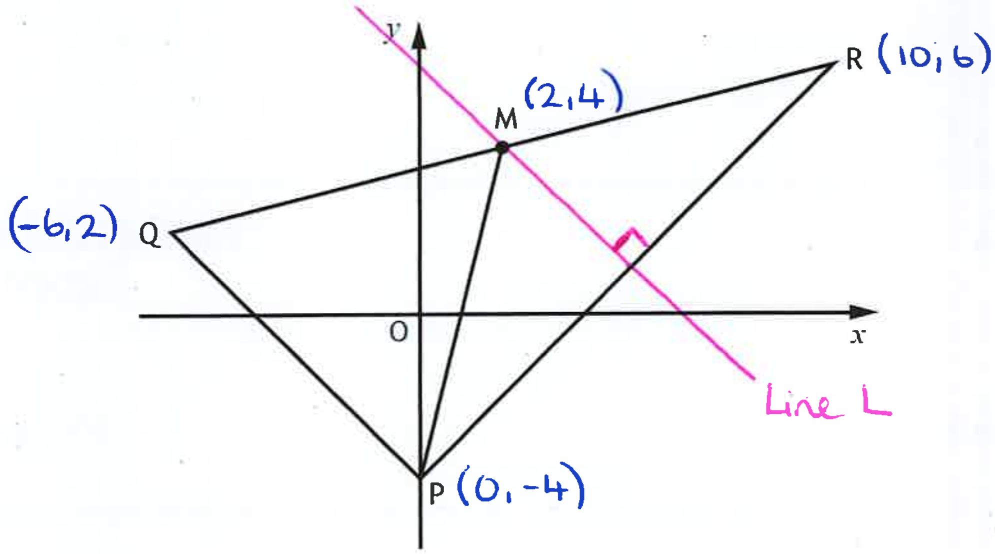
THURSDAY, 12 MAY
10:30 AM – 12:00 NOON

70 marks

2016 PAPER 2 – WORKED SOLUTIONS

HWallace

1. PQR is a triangle with vertices $P(0,-4)$, $Q(-6,2)$ and $R(10,6)$.



(a) (i) State the coordinates of M, the midpoint of QR. **1**

(ii) Hence find the equation of PM, the median through P. **2**

(b) Find the equation of the line, *L*, passing through M and perpendicular to PR. **3**

(c) Show that line *L* passes through the midpoint of PR. **3**

a) i) Midpoint $M\left(\frac{-6+10}{2}, \frac{2+6}{2}\right)$

$M(2, 4)$ ✓

ii) Median through P.

Gradient

$m_{PM} = \frac{4-(-4)}{2-0} = \frac{8}{2} = 4$ ✓

Equation $y = mx + c$

$y = 4x - 4$ ✓

1) b) Gradient of PR

$$m_{PR} = \frac{6-(-4)}{10-0} = \frac{10}{10} = 1$$

Perpendicular gradient

$m_{\perp} = -1$ since $m_1 \times m_2 = -1$

Equation of L passing $(2,4)$

$$y - b = m(x-a)$$
$$y - 4 = -1(x-2)$$
$$y - 4 = -x + 2$$
$$y = -x + 6$$

c) Consider midpoint of PR

Midpoint $\left(\frac{0+10}{2}, \frac{-4+6}{2}\right)$

Midpoint $(5, 1)$

Check substitution into line L

If $x = 5$, $y = -5 + 6$
$y = 1$ as required.

↳ Line L does pass $(5,1)$, the midpoint of PR.

2. Find the range of values for p such that $x^2 - 2x + 3 - p = 0$ has no real roots. 3

$b^2 - 4ac < 0$ for no real roots

$a = 1$

$b = -2$

$c = 3 - p$

$(-2)^2 - 4(1)(3-p) < 0$

$4 - 12 + 4p < 0$

$4p - 8 < 0$

$4p < 8$

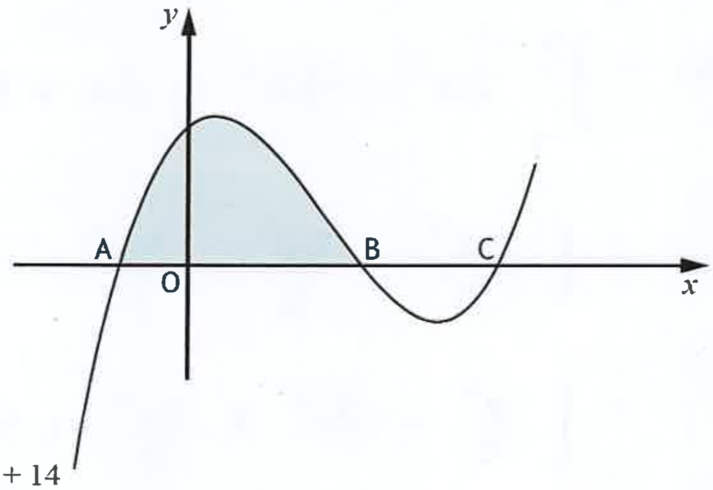
$p < 2$

3. (a) (i) Show that $(x+1)$ is a factor of $2x^3 - 9x^2 + 3x + 14$. **2**

(ii) Hence solve the equation $2x^3 - 9x^2 + 3x + 14 = 0$. **3**

(b) The diagram below shows the graph with equation $y = 2x^3 - 9x^2 + 3x + 14$.

The curve cuts the x-axis at A, B and C.



$y = 2x^3 - 9x^2 + 3x + 14$

(i) Write down the coordinates of the points A and B. **1**

(ii) Hence calculate the shaded area in the diagram. **4**

a) i) Check factor $(x+1)$

$$\begin{array}{r|rrrr} -1 & 2 & -9 & 3 & 14 \\ & \downarrow & -2 & 11 & -14 \\ \hline & 2 & -11 & 14 & \boxed{0} \end{array}$$

Since remainder is zero, $x = -1$ is a root and $(x+1)$ is a factor.

ii) $2x^3 - 9x^2 + 3x + 14 = 0$

$(x+1)(2x^2 - 11x + 14) = 0$

$(x+1)(2x-7)(x-2) = 0$

$x+1=0$ $\quad$ $2x-7=0$ $\quad$ $x-2=0$

$x=-1$ $\quad$ $x = 7/2$ $\quad$ $x=2$

↳ $x = -1, 2, 3\tfrac{1}{2}$.

3b) i) From roots in part a) ii)

A$(-1,0)$ and B$(2,0)$

ii) Area $= \int_{-1}^{2} 2x^3 - 9x^2 + 3x + 14 \, . dx$

$$= \left[\frac{2x^4}{4} - \frac{9x^3}{3} + \frac{3x^2}{2} + 14x \right]_{-1}^{2}$$

$$= \left[\frac{x^4}{2} - 3x^3 + \frac{3x^2}{2} + 14x \right]_{-1}^{2}$$

$$= \left(\frac{(2)^4}{2} - 3(2)^3 + \frac{3(2)^2}{2} + 14(2) \right)$$

$$- \left(\frac{(-1)^4}{2} - 3(-1)^3 + \frac{3(-1)^2}{2} + 14(-1) \right)$$

$$= (8 - 24 + 6 + 28) - \left(\frac{1}{2} + 3 + \frac{3}{2} - 14 \right)$$

$$= 18 - (-9)$$

Area $= 27$ units^2.

4. Circles C_1 and C_2 have equations $(x+5)^2+(y-6)^2=9$ and $x^2+y^2-6x-16=0$ respectively.

(a) Write down the centres and radii of C_1 and C_2. **4**

(b) Show that C_1 and C_2 do not intersect. **3**

a) Circle C_1

$$(x+5)^2+(y-6)^2=9$$

$$(x-a)^2+(y-b)^2=r^2$$

$a=-5 \quad b=6 \quad r^2=9$

$r=3$

$C_1(-5,6) \quad r_1=3$

Circle C_2

$$x^2+y^2-6x-16=0$$

$$x^2+y^2+2gx+2fy+c=0$$

$2g=-6 \quad 2f=0 \quad c=-16$

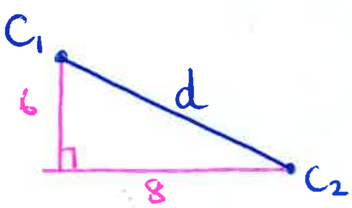
$g=-3 \quad f=0$

$r_2=\sqrt{(-3)^2+0-(-16)}$

$r_2=\sqrt{9+16}=\sqrt{25}=5$

$C_2(3,0) \quad r_2=5$

b) Distance between centres.



$d=10$

Pythagorean triple

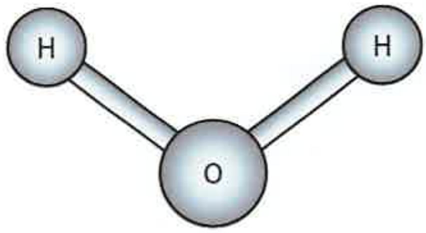
$r_1+r_2=3+5=8$

Since $d>r_1+r_2$

$10>8$

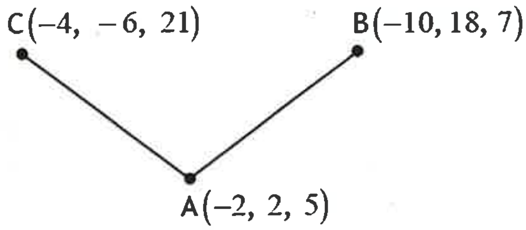
then the circles do not intersect.

5. The picture shows a model of a water molecule.



Relative to suitable coordinate axes, the oxygen atom is positioned at point A$(-2, 2, 5)$.

The two hydrogen atoms are positioned at points B$(-10, 18, 7)$ and C$(-4, -6, 21)$ as shown in the diagram below.



(a) Express $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in component form. **2**

(b) Hence, or otherwise, find the size of angle BAC. **4**

a) $\overrightarrow{AB} = b - a$

$= \begin{pmatrix} -10 \\ 18 \\ 7 \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \\ 5 \end{pmatrix}$

$\overrightarrow{AB} = \begin{pmatrix} -8 \\ 16 \\ 2 \end{pmatrix}$ ✓

$\overrightarrow{AC} = c - a$

$= \begin{pmatrix} -4 \\ -6 \\ 21 \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \\ 5 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} -2 \\ -8 \\ 16 \end{pmatrix}$ ✓

b) $|\overrightarrow{AB}| = \sqrt{(-8)^2 + 16^2 + 2^2}$

$= \sqrt{324}$

$|\overrightarrow{AB}| = 18$

$|\overrightarrow{AC}| = \sqrt{(-2)^2 + (-8)^2 + 16^2}$

$= \sqrt{324}$

$|\overrightarrow{AC}| = 18$ ✓

5) b) $\cos BAC = \dfrac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}|\,|\vec{AC}|}$

$$\vec{AB} \cdot \vec{AC} = (-8)(-2) + 16(-8) + 2(16)$$
$$= 16 - 128 + 32$$
$$\vec{AB} \cdot \vec{AC} = -80$$

$$\hookrightarrow \cos BAC = \frac{-80}{(18)(18)}$$

$$\cos BAC = -\frac{80}{324}$$

$$\cos BAC = -\frac{20}{81}$$

$$BAC = \cos^{-1}\left(-\frac{20}{81}\right)$$

$$BAC = 104.2949...$$

Angle $BAC = 104.3^\circ$ (1dp)

6. Scientists are studying the growth of a strain of bacteria. The number of bacteria present is given by the formula

$$B(t) = 200e^{0 \cdot 107t},$$

where t represents the number of hours since the study began.

(a) State the number of bacteria present at the start of the study. **1**

(b) Calculate the time taken for the number of bacteria to double. **4**

a) At the start, $t = 0$.

$B(0) = 200e^{0.107(0)}$

$B(0) = 200e^{0}$

$B(0) = 200$ bacteria

b) For Bacteria to double, $B(t) = 400$.

$200e^{0.107t} = 400$

$e^{0.107t} = 2$

$\ln e^{0.107t} = \ln 2$

$0.107t \cancel{\ln e} = \ln 2$

$0.107t = \ln 2$

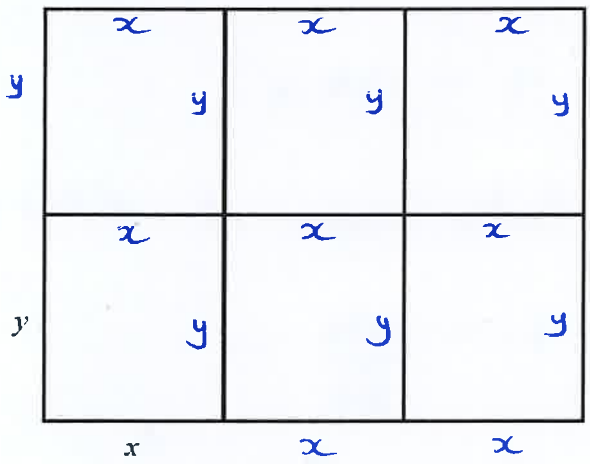
$t = \dfrac{\ln 2}{0.107}$

$t = 6.478011\ldots$

It will take 6.478 hours for bacteria to double.

7. A council is setting aside an area of land to create six fenced plots where local residents can grow their own food.

Each plot will be a rectangle measuring x metres by y metres as shown in the diagram.



(a) The area of land being set aside is 108 m^2.

Show that the total length of fencing, L metres, is given by

$$L(x) = 9x + \frac{144}{x}.$$ **3**

(b) Find the value of x that minimises the length of fencing required. **6**

a) Total length $= 9x + 8y$

Total Area $= 3x \times 2y = 6xy$

Since Area is 108m^2, then $6xy = 108$

$xy = 18$

$y = \frac{18}{x}$

Length, $L(x) = 9x + 8\left(\frac{18}{x}\right)$

$L(x) = 9x + \frac{144}{x}$ as required.

7) b) $$L(x) = 9x + \frac{144}{x}$$

$$L(x) = 9x + 144x^{-1}$$

$$L'(x) = 9 - 144x^{-2}$$

Stationary Points occur at $L'(x) = 0$

$$0 = 9 - \frac{144}{x^2}$$

$$\frac{144}{x^2} = 9$$

$$144 = 9x^2$$

$$16 = x^2$$

$x = \pm 4$ but $x > 0$ $\therefore$ $x = 4$ m only

Nature Table

x	$\xrightarrow{3}$	4	$\xrightarrow{5}$
$L'(x)$	$-$	0	$+$
slope	\	—	/

$$L'(3) = 9 - \frac{144}{3^2} = -7$$

$$L'(5) = 9 - \frac{144}{5^2} = 3.24$$

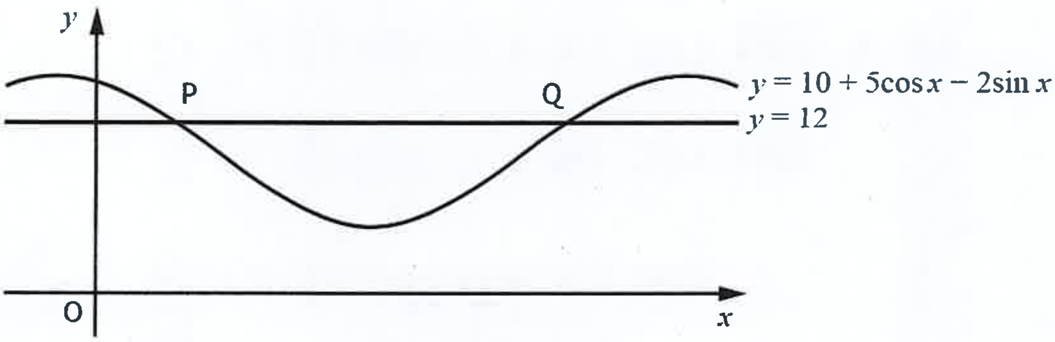
A minimum occurs when $x = 4$ m.

The length of fencing minimises when $x = 4$m.

8. (a) Express $5\cos x - 2\sin x$ in the form $k\cos(x + a)$, where $k > 0$ and $0 < a < 2\pi$. **4**

(b) The diagram shows a sketch of part of the graph of $y = 10 + 5\cos x - 2\sin x$ and the line with equation $y = 12$.

The line cuts the curve at the points P and Q.



Find the x-coordinates of P and Q. **4**

a) $k\cos(x + a) = k\cos x\cos a - k\sin x\sin a$

$= k\cos a.\cos x - k\sin a.\sin x$

$\rightarrow \quad 5.\cos x \quad - \quad 2.\sin x$

$k\cos a = 5$

$k\sin a = 2$

$k = \sqrt{5^2 + 2^2}$

$k = \sqrt{29}$

$\tan a = \dfrac{k\sin a}{k\cos a}$

$\tan a = \dfrac{2}{5}$

S	A
T	C

$ra = \tan^{-1}(2/5) = 0.3805$

$a = 0.3805$

$\hookrightarrow \underline{5\cos x - 2\sin x = \sqrt{29}\cos(x + 0.3805)}$

8) b) For $y = 10 + 5\cos x - 2\sin x$

$$y = 10 + \sqrt{29}\cos(x + 0.3805)$$

Solving for P and Q, let $y = y$

$$10 + \sqrt{29}\cos(x + 0.3805) = 12$$

$$\sqrt{29}\cos(x + 0.3805) = 2$$

$$\cos(x + 0.3805) = \frac{2}{\sqrt{29}}$$

$$ra = \cos^{-1}\left(\frac{2}{\sqrt{29}}\right) = 1.19028995\ldots$$

$\hookrightarrow$ $x + 0.3805 = 1.1902\ldots,\ 2\pi - 1.1902\ldots$

$$x + 0.3805 = 1.1902\ldots,\ 5.0928\ldots$$

$$x = 0.8097\ldots,\ 4.7123\ldots$$

$\hookrightarrow$ $P(0.810, 12)$ $Q(4.712, 12)$

9. For a function f, defined on a suitable domain, it is known that:

- $f'(x)=\dfrac{2x+1}{\sqrt{x}} = \dfrac{2x+1}{x^{1/2}} = \dfrac{2x}{x^{1/2}} + \dfrac{1}{x^{1/2}} = 2x^{1/2}+x^{-1/2}$ ✓
- $f(9)=40$

Express $f(x)$ in terms of x. **4**

$$f(x) = \int f'(x)\,.\,dx$$

$$f(x) = \int 2x^{1/2} + x^{-1/2}\,.\,dx$$

$$f(x) = \frac{2x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + c$$ ✓

$$f(x) = \frac{4}{3}\sqrt{x^3} + 2\sqrt{x} + c$$ ✓

When $f(9) = 40$

$$\frac{4}{3}\sqrt{9^3} + 2\sqrt{9} + c = 40$$

$$\frac{4}{3}(27) + 2(3) + c = 40$$

$$36 + 6 + c = 40$$

$$42 + c = 40$$

$$c = -2$$

↳ $f(x) = \frac{4}{3}\sqrt{x^3} + 2\sqrt{x} - 2$ ✓

10. (a) Given that $y=\left(x^2+7\right)^{\frac{1}{2}}$, find $\frac{dy}{dx}$. **2**

(b) Hence find $\int \frac{4x}{\sqrt{x^2+7}}\,dx$. **1**

a) $y = (x^2+7)^{1/2}$

$\frac{dy}{dx} = \frac{1}{2}(x^2+7)^{-1/2} \times 2x$

$\frac{dy}{dx} = x(x^2+7)^{-1/2}$

b) $\int \frac{4x}{\sqrt{x^2+7}} \cdot dx$

$= \int 4x(x^2+7)^{-1/2} \cdot dx$

$= 4\int x(x^2+7)^{-1/2} \cdot dx$

$= 4(x^2+7)^{1/2} + c$

11. (a) Show that $\sin 2x \tan x = 1 - \cos 2x$, where $\frac{\pi}{2} < x < \frac{3\pi}{2}$. **4**

(b) Given that $f(x) = \sin 2x \tan x$, find $f'(x)$. **2**

a) $LHS = \sin 2x \tan x$

$\sin 2x = \sin x \cos x$

$LHS = 2\sin x \cos x \tan x$

$\tan x = \frac{\sin x}{\cos x}$

$LHS = 2\sin x \cos x \left(\frac{\sin x}{\cos x}\right)$

$LHS = 2\sin^2 x$

$\cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

$LHS = 1 - \cos 2x$

$LHS = RHS$ as required.

b) $f(x) = \sin 2x \tan x$

$f(x) = 1 - \cos 2x$

$f'(x) = 2\sin 2x.$