H

National
Qualifications
2016

X747/76/11

Mathematics
Paper 1
(Non-Calculator)

THURSDAY, 12 MAY

9:00 AM – 10:10 AM

60 marks

2016 PAPER 1 – WORKED SOLUTIONS

HWallace

1. Find the equation of the line passing through the point $(-2, 3)$ which is parallel to the line with equation $y + 4x = 7$. **2**

Parallel to $y + 4x = 7$

$y = -4x + 7$

$m = -4$ ✓

Equation $y - b = m(x - a)$

$m = -4$ $\quad y - 3 = -4(x + 2)$

$a = -2$

$b = 3$ $\quad y - 3 = -4x - 8$

$y = -4x - 5$ ✓

2. Given that $y = 12x^3 + 8\sqrt{x}$, where $x > 0$, find $\frac{dy}{dx}$. **3**

$y = 12x^3 + 8x^{1/2}$ ✓

$\frac{dy}{dx} = 36x^2 + 4x^{-1/2}$ ✓✓

$\frac{dy}{dx} = 36x^2 + \frac{4}{\sqrt{x}}$

3. A sequence is defined by the recurrence relation $u_{n+1} = \frac{1}{3}u_n + 10$ with $u_3 = 6$.

(a) Find the value of u_4. **1**

(b) Explain why this sequence approaches a limit as $n \to \infty$. **1**

(c) Calculate this limit. **2**

a) $U_4 = \frac{1}{3}U_3 + 10$

$U_4 = \frac{1}{3}(6) + 10$

$U_4 = 2 + 10$

$U_4 = 12$ ✓

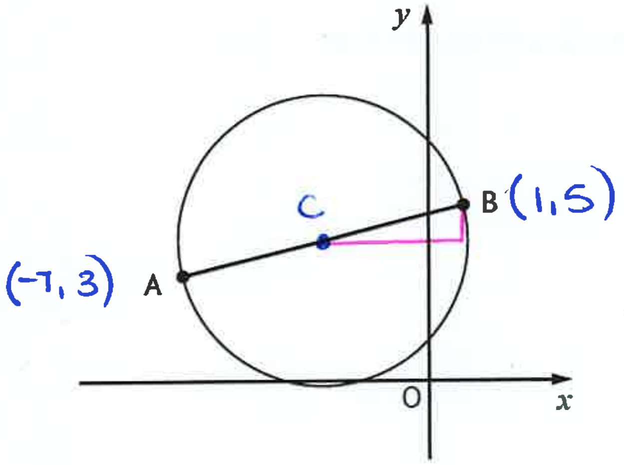
b) A limit exists for this linear recorrence relation since $-1 < \frac{1}{3} < 1$. ✓

c) $L = \frac{10}{1-\frac{1}{3}}$ ✓ $= \frac{10}{\frac{2}{3}} = \frac{30}{2} = 15$

The limit is 15. ✓

4. A and B are the points $(-7, 3)$ and $(1, 5)$.

AB is a diameter of a circle.



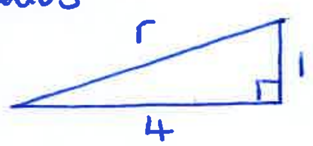
Find the equation of this circle. **3**

Centre C is the midpoint of AB

$C\left(\frac{-7+1}{2}, \frac{3+5}{2}\right)$

$C(-3, 4)$ ✓

Radius



$r^2 = 1^2 + 4^2$

$r^2 = 1 + 16$

$r^2 = 17$

$r = \sqrt{17}$ ✓

Equation:

$(x-a)^2 + (y-b)^2 = r^2$

$(x+3)^2 + (y-4)^2 = 17$ ✓

5. Find $\int 8\cos(4x+1)\,dx$. **2**

$= \frac{8}{4}\sin(4x+1) + c$

$= 2\sin(4x+1) + c$

6. Functions f and g are defined on $\mathbb{R}$, the set of real numbers.

The inverse functions f^{-1} and g^{-1} both exist.

(a) Given $f(x) = 3x + 5$, find $f^{-1}(x)$. **3**

(b) If $g(2) = 7$, write down the value of $g^{-1}(7)$. **1**

a) Let $y = 3x + 5$

$3x = y - 5$

$x = \frac{y-5}{3}$

$\hookrightarrow f^{-1}(x) = \frac{x-5}{3}$

b) If $g(2) = 7$

then $g^{-1}(7) = 2.$

7. Three vectors can be expressed as follows:

$$\overrightarrow{FG} = -2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$$

$$\overrightarrow{GH} = 3\mathbf{i} + 9\mathbf{j} - 7\mathbf{k}$$

$$\overrightarrow{EH} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

(a) Find $\overrightarrow{FH}$. **2**

(b) Hence, or otherwise, find $\overrightarrow{FE}$. **2**

a) $\overrightarrow{FH} = \overrightarrow{FG} + \overrightarrow{GH}$ ✓

$$\overrightarrow{FH} = \begin{pmatrix} -2 \\ -6 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 9 \\ -7 \end{pmatrix}$$

$\overrightarrow{FH} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$ or $\overrightarrow{FH} = i + 3j - 4k$ ✓

b) $\overrightarrow{FE} = \overrightarrow{FH} - \overrightarrow{EH}$ ✓

$$\overrightarrow{FE} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$\overrightarrow{FE} = \begin{pmatrix} -1 \\ 0 \\ -5 \end{pmatrix}$ or $\overrightarrow{FE} = -i - 5k$ ✓

8. Show that the line with equation $y = 3x - 5$ is a tangent to the circle with equation $x^2 + y^2 + 2x - 4y - 5 = 0$ and find the coordinates of the point of contact. **5**

Sub in $y = 3x - 5$

$x^2 + (3x-5)^2 + 2x - 4(3x-5) - 5 = 0$ ✓

$x^2 + 9x^2 - 30x + 25 + 2x - 12x + 20 - 5 = 0$

$10x^2 - 40x + 40 = 0$ ✓

$10(x^2 - 4x + 4) = 0$

$10(x-2)(x-2) = 0$

$10(x-2)^2 = 0$ ✓

As there is one repeated factor of $(x-2)$, then the line is a tangent to the circle (one point of contact). ✓

$x - 2 = 0$

$x = 2$

$y = 3(2) - 5$

$y = 6 - 5$

$y = 1$

Point of Tangency $(2, 1)$. ✓

9. (a) Find the x-coordinates of the stationary points on the graph with equation $y = f(x)$, where $f(x) = x^3 + 3x^2 - 24x$. **4**

(b) Hence determine the range of values of x for which the function f is strictly increasing. **2**

a) $f(x) = x^3 + 3x^2 - 24x$

$f'(x) = 3x^2 + 6x - 24$

Stationary Points occur when $f'(x) = 0$

$3x^2 + 6x - 24 = 0$

$3(x^2 + 2x - 8) = 0$

$3(x+4)(x-2) = 0$

$x + 4 = 0$ $\quad$ $x - 2 = 0$

$x = -4$ $\quad$ $x = 2$

Stationary Points at $x = -4, 2$.

b) Nature Table

x	$\xrightarrow{-5}$	-4	$\xrightarrow{0}$	2	$\xrightarrow{3}$
$f'(x)$	+	0	−	0	+
slope	/	—	\	—	/

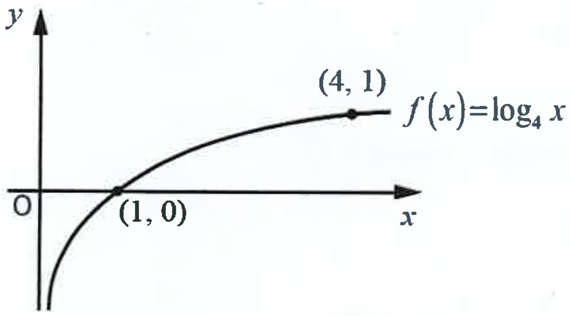
$f'(-5) = 3(-5)^2 + 6(-5) - 24 = (+)$
$= 75 - 30 - 24$

$f'(0) = 3(0)^2 + 6(0) - 24 = (-)$

$f'(3) = 3(3)^2 + 6(3) - 24 = (+)$
$= 27 + 18 - 24$

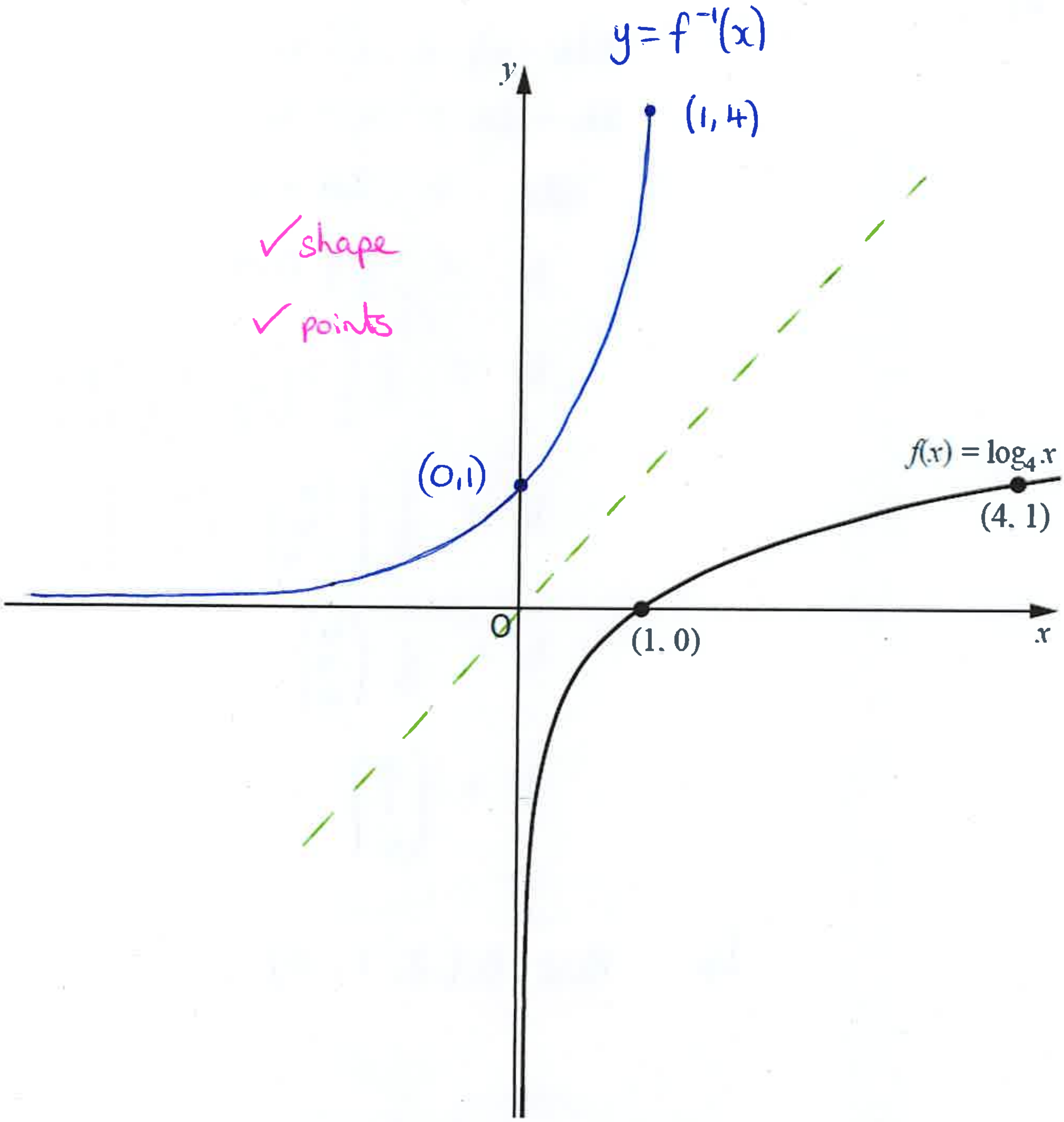
Function f is increasing when $x < -4$ and $x > 2$

10. The diagram below shows the graph of the function $f(x)=\log_4 x$, where $x>0$.



The inverse function, f^{-1}, exists.

On the diagram in your answer booklet, sketch the graph of the inverse function. **2**



11. (a) A and C are the points $(1, 3, -2)$ and $(4, -3, 4)$ respectively.

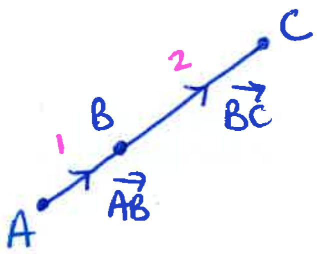
Point B divides AC in the ratio $1 : 2$.

Find the coordinates of B. **2**

(b) $k\overrightarrow{AC}$ is a vector of magnitude 1, where $k > 0$.

Determine the value of k. **3**

a)



$$\frac{\overrightarrow{AB}}{\overrightarrow{BC}} = \frac{1}{2}$$

$$2\overrightarrow{AB} = \overrightarrow{BC}$$

$$2(b - a) = c - b$$

$$2b - 2a = c - b$$

$$3b = 2a + c$$

$$b = \frac{1}{3}(2a + c) \checkmark$$

$$b = \frac{1}{3}\left[2\begin{pmatrix}1\\3\\-2\end{pmatrix} + \begin{pmatrix}4\\-3\\4\end{pmatrix}\right]$$

$$b = \frac{1}{3}\left[\begin{pmatrix}2\\6\\-4\end{pmatrix} + \begin{pmatrix}4\\-3\\4\end{pmatrix}\right]$$

$$b = \frac{1}{3}\begin{pmatrix}6\\3\\0\end{pmatrix}$$

$$b = \begin{pmatrix}2\\1\\0\end{pmatrix}$$

↳ Point B $(2, 1, 0)$ ✓

11) b) $k\vec{AC}$ has magnitude of 1.

$\therefore k|\vec{AC}| = 1$

$\vec{AC} = c - a$

$\vec{AC} = \begin{pmatrix} 4 \\ -3 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

$\vec{AC} = \begin{pmatrix} 3 \\ -6 \\ 6 \end{pmatrix}$ ✓

$|\vec{AC}| = \sqrt{3^2 + (-6)^2 + 6^2}$

$= \sqrt{9 + 36 + 36}$

$= \sqrt{81}$

$|\vec{AC}| = 9$ ✓

If $k|\vec{AC}| = 1$

$9k = 1$

$k = \frac{1}{9}$ ✓

12. The functions f and g are defined on $\mathbb{R}$, the set of real numbers by

$f(x) = 2x^2 - 4x + 5$ and $g(x) = 3 - x$.

(a) Given $h(x) = f(g(x))$, show that $h(x) = 2x^2 - 8x + 11$. **2**

(b) Express $h(x)$ in the form $p(x+q)^2 + r$. **3**

a) $h(x) = f(g(x))$

$= f(3-x)$

$= 2(3-x)^2 - 4(3-x) + 5$

$= 2(9 - 6x + x^2) - 12 + 4x + 5$

$= 18 - 12x + 2x^2 - 12 + 4x + 5$

$h(x) = 2x^2 - 8x + 11$ as required.

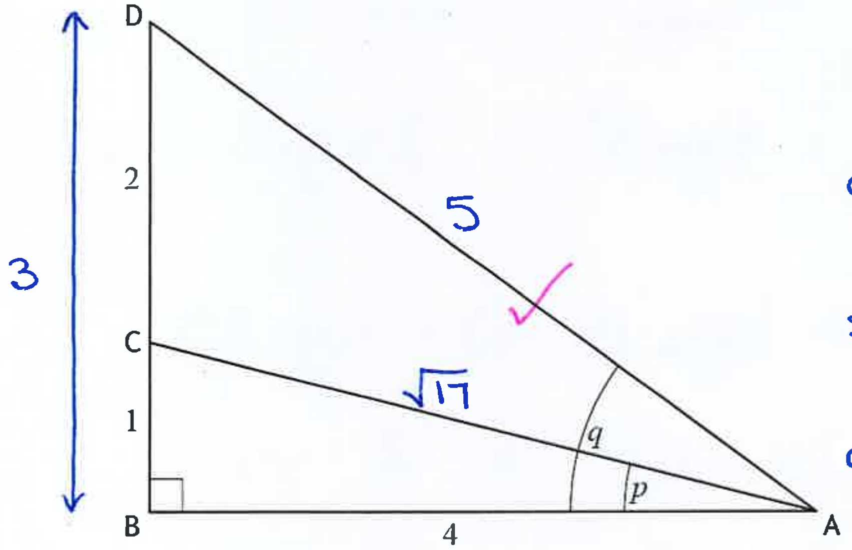
b) $h(x) = 2[x^2 - 4x] + 11$

$= 2[(x-2)^2 - 4] + 11$

$= 2(x-2)^2 - 8 + 11$

$h(x) = 2(x-2)^2 + 3$

13. Triangle ABD is right-angled at B with angles BAC $= p$ and BAD $= q$ and lengths as shown in the diagram below.



$\sin q = \frac{3}{5}$

$\cos q = \frac{4}{5}$

$\sin p = \frac{1}{\sqrt{17}}$

$\cos p = \frac{4}{\sqrt{17}}$

Show that the exact value of $\cos(q-p)$ is $\frac{19\sqrt{17}}{85}$. **5**

$$\cos(q-p) = \cos q \cos p + \sin q \sin p$$

$$= \left(\frac{4}{5}\right)\left(\frac{4}{\sqrt{17}}\right) + \left(\frac{3}{5}\right)\left(\frac{1}{\sqrt{17}}\right)$$

$$= \frac{16}{5\sqrt{17}} + \frac{3}{5\sqrt{17}}$$

$$= \frac{19}{5\sqrt{17}} \times \frac{\sqrt{17}}{\sqrt{17}}$$

$$\cos(q-p) = \frac{19\sqrt{17}}{85}$$ as required

14. (a) Evaluate $\log_5 25$. **1**

(b) Hence solve $\log_4 x + \log_4 (x-6) = \log_5 25$, where $x > 6$. **5**

a) $\log_5 25 = \log_5 5^2 = 2\log_5 5 = \underline{\underline{2}}$ ✓

b) $\log_4 x + \log_4 (x-6) = \log_5 25$

$\log_4 x(x-6) = 2$ ✓✓

$\log_4 x(x-6) = 2\log_4 4$

$\log_4 x(x-6) = \log_4 4^2$

$\log_4 x(x-6) = \log_4 16$

$x(x-6) = 16$ ✓

$x^2 - 6x = 16$

$x^2 - 6x - 16 = 0$ ✓

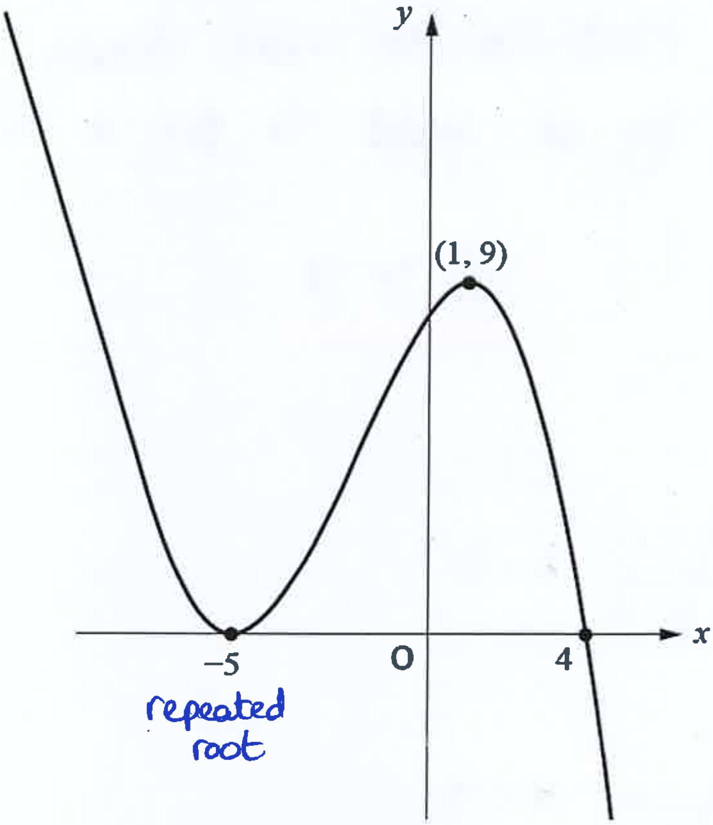
$(x+2)(x-8) = 0$

$\cancel{x+2 = 0}$ $\qquad$ $x - 8 = 0$

$\cancel{x = -2}$ $\qquad$ $\underline{\underline{x = 8}}$ ✓

No solution as $x > 6$

15. The diagram below shows the graph with equation $y = f(x)$, where $f(x) = k(x-a)(x-b)^2$.



(a) Find the values of a, b and k. **3**

(b) For the function $g(x) = f(x) - d$, where d is positive, determine the range of values of d for which $g(x)$ has exactly one real root. **1**

a) $y = k(x-a)(x-b)^2$

$y = k(x-4)(x+5)^2$

At $(1,9)$

$9 = k(1-4)(1+5)^2$

$9 = k(-3)(36)$

$-3 = 36k$

$k = -\frac{1}{12}$ with $a = 4$ and $b = -5$

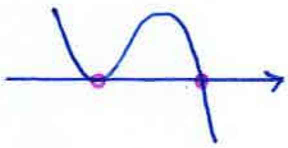
15) b) For $g(x) = f(x) - d$ where $d > 0$

$f(x)$ has to move down

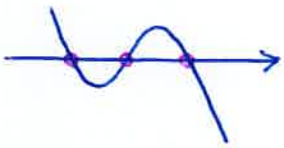
by at least 9 for 1 root.

$d > 9$ ✓

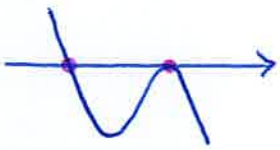
Think:



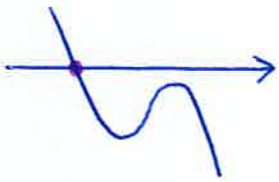
$d = 0$



$d < 9$



$d = 9$



$d > 9$ only one real root!