H

National
Qualifications
2015

X747/76/12

Mathematics
Paper 2

WEDNESDAY, 20 MAY
10:30 AM – 12:00 NOON

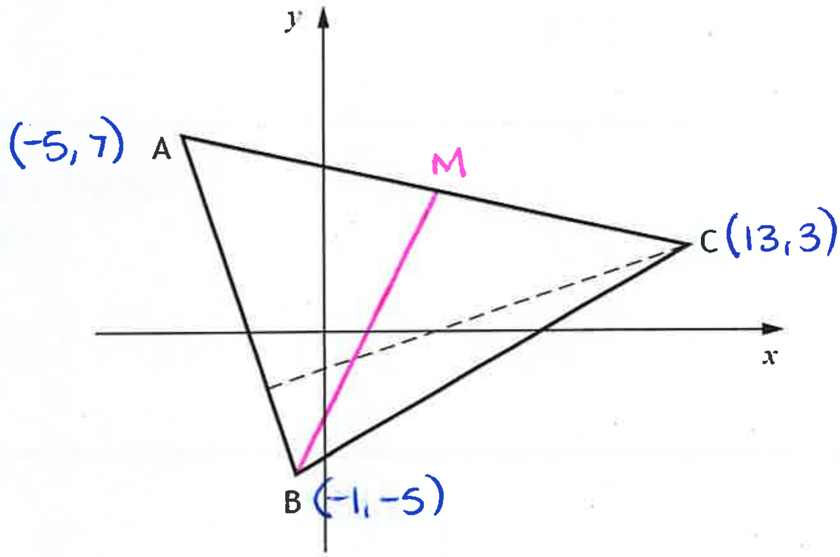
70 marks

2015 PAPER 2 – WORKED SOLUTIONS

HWallace

1. The vertices of triangle ABC are A(−5, 7), B(−1, −5) and C(13, 3) as shown in the diagram.

The broken line represents the altitude from C.



(a) Show that the equation of the altitude from C is $x - 3y = 4$. **4**

(b) Find the equation of the median from B. **3**

(c) Find the coordinates of the point of intersection of the altitude from C and the median from B. **2**

a) Altitude from C

$m_{AB} = \frac{-5-7}{-1-(-5)} = \frac{-12}{4} = -3$ ✓

$m_{\perp} = \frac{1}{3}$ since $m_1 \times m_2 = -1$ ✓

Equation C(13, 3)

$y - b = m(x - a)$

$y - 3 = \frac{1}{3}(x - 13)$ ✓

$3y - 9 = x - 13$

$3y = x - 4$

↳ $x - 3y = 4$ ✓

1) b) Median from B.

Midpoint $M\left(\frac{-5+13}{2}, \frac{7+3}{2}\right)$

$M(4, 5)$ ✓

Gradient

$m_{BM} = \frac{5-(-5)}{4-(-1)} = \frac{10}{5} = 2$ ✓

Equation

$$y - b = m(x - a)$$
$$y - 5 = 2(x - 4)$$
$$y - 5 = 2x - 8$$
$$y = 2x - 3$$ ✓

c) Point of intersection

$3y = x - 4$

and $y = 2x - 3$

$3y = 6x - 9$

Let $3y = 3y$

$x - 4 = 6x - 9$

$5 = 5x$

$x = 1$ ✓

Now $y = 2(1) - 3$

$y = -1$ ✓

Point of intersection $(1, -1)$

2. Functions f and g are defined on suitable domains by

$$f(x) = 10 + x \quad \text{and} \quad g(x) = (1 + x)(3 - x) + 2.$$

(a) Find an expression for $f(g(x))$. **2**

(b) Express $f(g(x))$ in the form $p(x + q)^2 + r$. **3**

(c) Another function h is given by $h(x) = \dfrac{1}{f(g(x))}$.

What values of x cannot be in the domain of h? **2**

a) $f(g(x)) = f((1+x)(3-x)+2)$ ✓

$= 10 + (1+x)(3-x) + 2$ ✓

$f(g(x)) = 12 + (1+x)(3-x)$

b) $f(g(x)) = 3 - x + 3x - x^2 + 12$

$f(g(x)) = -x^2 + 2x + 15$ ✓

$= -1[x^2 - 2x] + 15$ ✓

$= -1[(x-1)^2 - 1] + 15$

$= -(x-1)^2 + 1 + 15$

$f(g(x)) = -(x-1)^2 + 16$ ✓

c) For $h(x) = \dfrac{1}{f(g(x))}$, we cannot divide by zero.

$f(g(x)) \neq 0$ ✓

$-(x-1)^2 + 16 \neq 0$

$(x-1)^2 \neq 16$

$(x-1) \neq \sqrt{16}$

$x - 1 \neq \pm 4$

$x - 1 \neq -4$ $\quad$ $x - 1 \neq 4$

$x \neq -3$ $\quad$ $x \neq 5$

$x \neq -3, 5$ ✓

3. A version of the following problem first appeared in print in the 16th Century.

A frog and a toad fall to the bottom of a well that is 50 feet deep.

Each day, the frog climbs 32 feet and then rests overnight. During the night, it slides down $\frac{2}{3}$ of its height above the floor of the well.

The toad climbs 13 feet each day before resting.

Overnight, it slides down $\frac{1}{4}$ of its height above the floor of the well.

Their progress can be modelled by the recurrence relations:

- $f_{n+1} = \frac{1}{3}f_n + 32, \qquad f_1 = 32$
- $t_{n+1} = \frac{3}{4}t_n + 13, \qquad t_1 = 13$

where f_n and t_n are the heights reached by the frog and the toad at the end of the nth day after falling in.

(a) Calculate t_2, the height of the toad at the end of the second day. **1**

(b) Determine whether or not either of them will eventually escape from the well. **5**

a) $t_2 = \frac{3}{4}t_1 + 13 = \frac{3}{4}(13) + 13 = 22.75$ feet ✓

b) Limit of Frog

$L = \frac{32}{1 - 1/3}$ ✓

$L = \frac{32}{2/3}$

$L = 48$ ✓

Limit of Toad

$L = \frac{13}{1 - 3/4}$ ✓

$L = \frac{13}{1/4}$

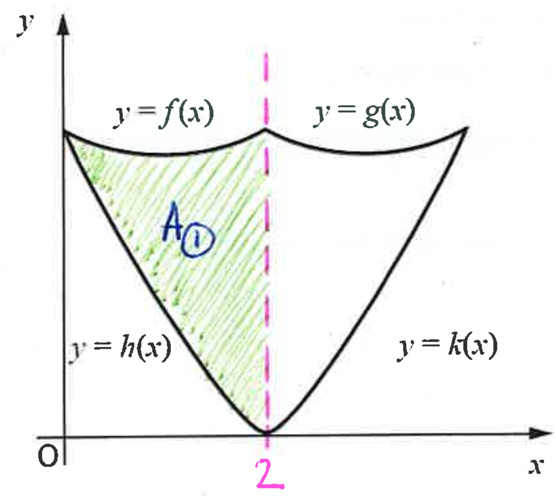
$L = 52$ ✓

Since $48 < 50$, the frog will not escape.

But since $52 > 50$, the toad will escape the well! ✓

4. A wall plaque is to be made to commemorate the 150th anniversary of the publication of "*Alice's Adventures in Wonderland*".

The edges of the wall plaque can be modelled by parts of the graphs of four quadratic functions as shown in the sketch.



- $f(x) = \frac{1}{4}x^2 - \frac{1}{2}x + 3$
- $g(x) = \frac{1}{4}x^2 - \frac{3}{2}x + 5$
- $h(x) = \frac{3}{8}x^2 - \frac{9}{4}x + 3$
- $k(x) = \frac{3}{8}x^2 - \frac{3}{4}x$

(a) Find the x-coordinate of the point of intersection of the graphs with equations $y = f(x)$ and $y = g(x)$. **2**

The graphs of the functions $f(x)$ and $h(x)$ intersect on the y-axis.

The plaque has a vertical line of symmetry.

(b) Calculate the area of the wall plaque. **7**

a) For point of intersection, let y=y

$\frac{1}{4}x^2 - \frac{1}{2}x + 3 = \frac{1}{4}x^2 - \frac{3}{2}x + 5$ ✓

$x = 2$ ✓

4) b) Area $A_{①} = \int_0^2 f(x) - h(x) \, . \, dx$

$$= \int_0^2 \left(\frac{1}{4}x^2 - \frac{1}{2}x + 3\right) - \left(\frac{3}{8}x^2 - \frac{9}{4}x + 3\right) . dx$$

$$= \int_0^2 \frac{1}{4}x^2 - \frac{1}{2}x + 3 - \frac{3}{8}x^2 + \frac{9}{4}x - 3 \, . \, dx$$

$$= \int_0^2 \frac{2}{8}x^2 - \frac{2}{4}x - \frac{3}{8}x^2 + \frac{9}{4}x \, . \, dx$$

$$= \int_0^2 \frac{7}{4}x - \frac{1}{8}x^2 \, . \, dx$$

$$= \left[\frac{7x^2}{8} - \frac{x^3}{24}\right]_0^2$$

$$= \left(\frac{7(2)^2}{8} - \frac{(2)^3}{24}\right) - \left(\frac{7(0)^2}{8} - \frac{(0)^3}{24}\right)$$

$$= \left(\frac{28}{8} - \frac{8}{24}\right) - 0$$

$$= \frac{7}{2} - \frac{1}{3}$$

$$= \frac{21}{6} - \frac{2}{6}$$

$$A_{①} = \frac{19}{6} \text{ units}^2$$

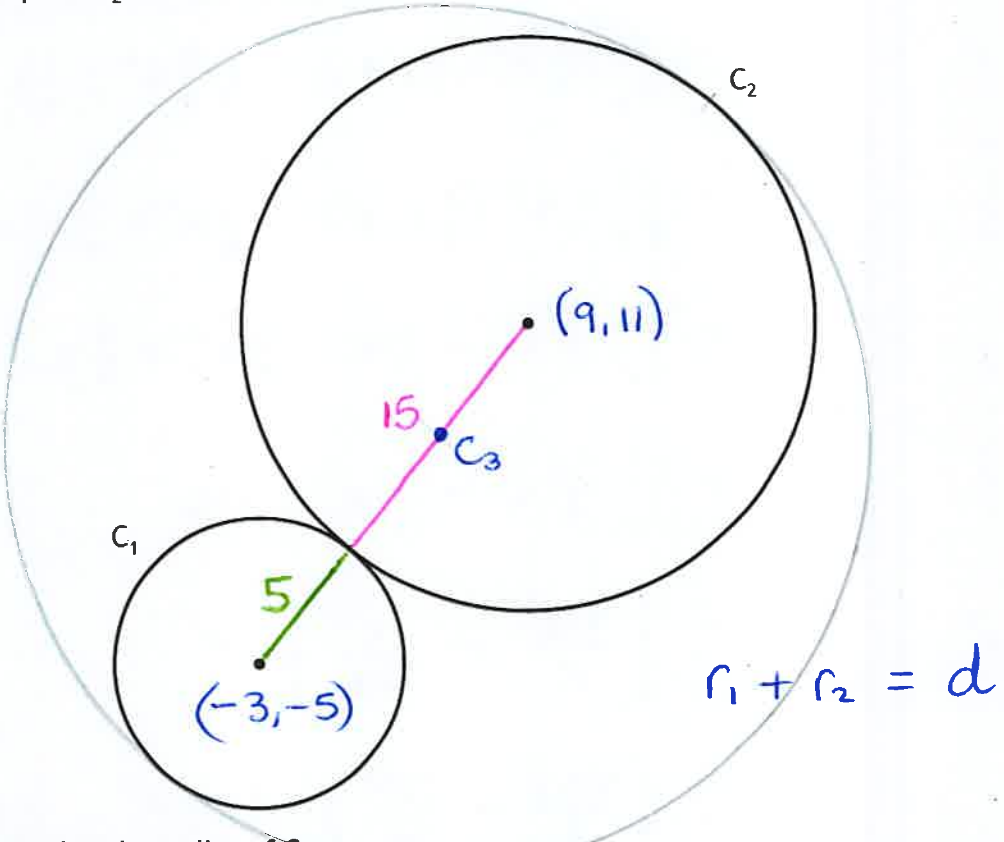
$$\text{Total Area} = \frac{19}{6} \times 2$$

$$\text{Total Area} = \frac{19}{3} = 6\frac{1}{3} \text{ units}^2$$

5. Circle C_1 has equation $x^2 + y^2 + 6x + 10y + 9 = 0$.

The centre of circle C_2 is $(9, 11)$.

Circles C_1 and C_2 touch externally.



(a) Determine the radius of C_2. **4**

A third circle, C_3, is drawn such that:

- both C_1 and C_2 touch C_3 internally
- the centres of C_1, C_2 and C_3 are collinear.

(b) Determine the equation of C_3. **4**

a) C_1 $\quad x^2 + y^2 + 6x + 10y + 9 = 0$

$x^2 + y^2 + 2gx + 2fy + c = 0$

$2g = 6 \quad 2f = 10 \quad c = 9$

$g = 3 \quad f = 5$

$C_1\,(-3, -5)$

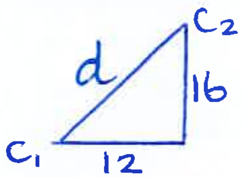
$r_1 = \sqrt{3^2 + 5^2 - 9}$

$r_1 = \sqrt{9 + 25 - 9}$

$r_1 = \sqrt{25}$

$r_1 = 5$ units

Distance between centres.



$d^2 = 12^2 + 16^2$

$d^2 = 400$

$d = 20$

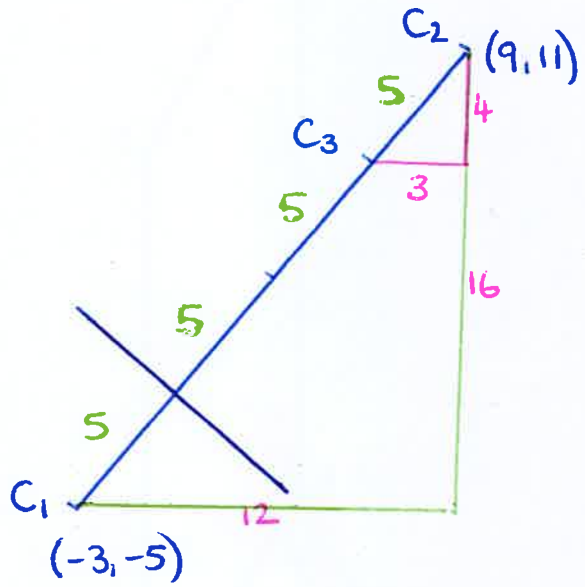
$r_2 = d - r_1$

$r_2 = 20 - 5$

$r_2 = 15$ units

5) b) For C_3, then diameter = 40 units.

and r_3 = 20 units. ✓



C_3 divides the line between C_1 and C_2 in the ratio 3 : 1. ✓

Centre $C_3(6, 7)$ ✓

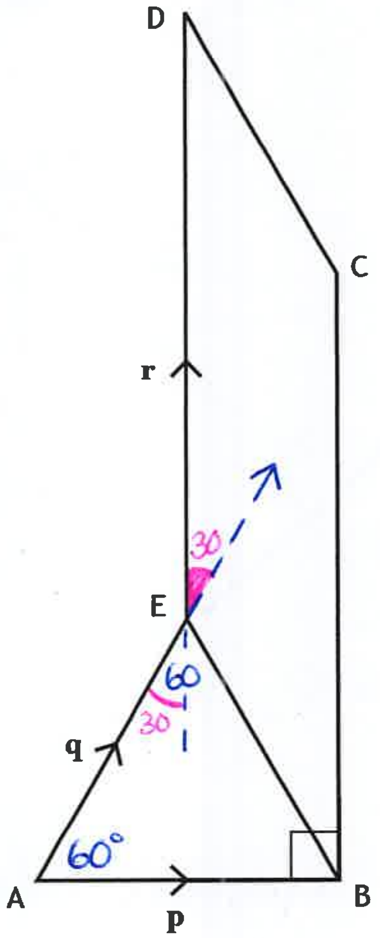
Equation $(x-a)^2 + (y-b)^2 = r^2$

$(x-6)^2 + (y-7)^2 = 20^2$

$(x-6)^2 + (y-7)^2 = 400$. ✓

6. Vectors **p**, **q** and **r** are represented on the diagram as shown.

- BCDE is a parallelogram
- ABE is an equilateral triangle $60°$
- $|\mathbf{p}| = 3$ and $|q| = 3$
- Angle ABC = 90°



$\underline{p}$ is perpendicular to $\underline{r}$

therefore $\underline{p} \cdot \underline{r} = 0$

(a) Evaluate $\mathbf{p}.(\mathbf{q}+\mathbf{r})$. **3**

(b) Express $\overrightarrow{EC}$ in terms of **p**, **q** and **r**. **1**

(c) Given that $\overrightarrow{AE}.\overrightarrow{EC} = 9\sqrt{3} - \frac{9}{2}$, find $|\mathbf{r}|$. **3**

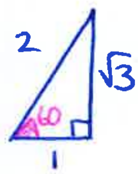
a) $p.(q + r) = p.q + p.r$ ✓

$= |p||q| \cos A + 0$ ✓

$= 3 \times 3 \times \cos 60°$

$= 9 \left(\frac{1}{2}\right)$

$p.(q+r) = 4\frac{1}{2}$ ✓



6) b) $\vec{EC} = \vec{EA} + \vec{AB} + \vec{BC}$

$\vec{EC} = -q + p + r$ ✓

$\vec{EC} = p - q + r$

6) c) $\vec{AE} \cdot \vec{EC} = 9\sqrt{3} - \frac{9}{2}$

$q \cdot (p - q + r) = 9\sqrt{3} - \frac{9}{2}$

$q \cdot p - q \cdot q + q \cdot r = 9\sqrt{3} - \frac{9}{2}$ ✓

think... $q \cdot p = p \cdot r = \frac{9}{2}$ from part (a)

$q \cdot q = |q|^2 = 3^2 = 9$

↳ $\frac{9}{2} - 9 + q \cdot r = 9\sqrt{3} - \frac{9}{2}$ ✓

$q \cdot r = 9\sqrt{3} - \frac{9}{2} - \frac{9}{2} + 9$

$q \cdot r = 9\sqrt{3}$

$|q||r| \cos 30° = 9\sqrt{3}$

$3 \times |r| \times \frac{\sqrt{3}}{2} = 9\sqrt{3}$

$3\sqrt{3}|r| = 18\sqrt{3}$

$|r| = 6$ units. ✓

7. (a) Find $\int (3\cos 2x + 1)\,dx$. **2**

(b) Show that $3\cos 2x + 1 = 4\cos^2 x - 2\sin^2 x$. **2**

(c) Hence, or otherwise, find $\int (\sin^2 x - 2\cos^2 x)\,dx$. **2**

a) $\int 3\cos 2x + 1 \,.\, dx = \frac{3}{2}\sin 2x + x + c$

b) $LHS = 3\cos 2x + 1$

$LHS = 3(\cos^2 x - \sin^2 x) + 1$

$LHS = 3\cos^2 x - 3\sin^2 x + \sin^2 x + \cos^2 x$

$LHS = 4\cos^2 x - 2\sin^2 x$

LHS = RHS as required

c) $\int \sin^2 x - 2\cos^2 x \,.\, dx.$

$= -\frac{1}{2}\int (4\cos^2 x - 2\sin^2 x)\,.\,dx$

$= -\frac{1}{2}\int (3\cos 2x + 1)\,.\,dx$

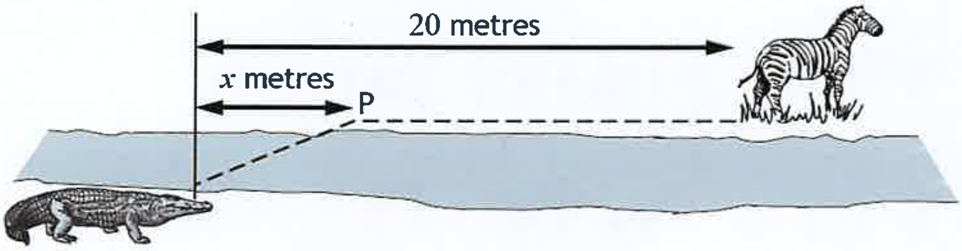
$= -\frac{1}{2}\left[\frac{3}{2}\sin 2x + x\right] + c$

$= -\frac{3}{4}\sin 2x - \frac{1}{2}x + c$

8. A crocodile is stalking prey located 20 metres further upstream on the opposite bank of a river.

Crocodiles travel at different speeds on land and in water.

The time taken for the crocodile to reach its prey can be minimised if it swims to a particular point, P, x metres upstream on the other side of the river as shown in the diagram.



The time taken, T, measured in tenths of a second, is given by

$$T(x) = 5\sqrt{36 + x^2} + 4(20 - x)$$

(a) (i) Calculate the time taken if the crocodile does not travel on land. **1**

(ii) Calculate the time taken if the crocodile swims the shortest distance possible. **1**

(b) Between these two extremes there is one value of x which minimises the time taken. Find this value of x and hence calculate the minimum possible time. **8**

a) i) If the crocodile only swims, then $x = 20\text{ m}$.

$T(20) = 5\sqrt{36+20^2} + 4(20-20)$

$= 5\sqrt{436} + 0$

$= 104.4$ tenths of a second. ✓

ii) If the crocodile swims the shortest distance, then $x = 0\text{ m}$.

$T(0) = 5\sqrt{36+0^2} + 4(20-0)$

$= 5\sqrt{36} + 80$

$= 30 + 80$

$= 110$ tenths of a second ✓

8) b) Optimisation to find x which minimises time.

$$T(x) = 5\sqrt{36 + x^2} + 4(20 - x)$$

$$T(x) = 5(36 + x^2)^{1/2} + 80 - 4x$$

$$T'(x) = \frac{5}{2}(36 + x^2)^{-1/2}(2x) - 4$$

$$T'(x) = 5x(36 + x^2)^{-1/2} - 4$$

$$T'(x) = \frac{5x}{\sqrt{36 + x^2}} - 4$$

Stationary Points occur when $T'(x) = 0$

$$0 = \frac{5x}{\sqrt{36 + x^2}} - 4$$

$$4\sqrt{36 + x^2} = 5x$$

$$\sqrt{36 + x^2} = \frac{5x}{4}$$

$$36 + x^2 = \frac{25x^2}{16}$$

$$576 + 16x^2 = 25x^2$$

$$576 = 9x^2$$

$$x^2 = 64$$

$$x = \pm 8$$

But $x > 0$ for distance

$\therefore x = 8$ metres.

Nature Table

x	$\rightarrow$ (7)	8	$\rightarrow$ (9)
$T'(x)$	$-$	0	$+$
slope	\	—	/

Time minimises when $x = 8$.

$$T(8) = 5\sqrt{36 + 8^2} + 4(20 - 8)$$

$$= 5\sqrt{100} + 48$$

$$= 50 + 48$$

$$= 98 \text{ tenths of a second}$$

This was a really tough question.

9. The blades of a wind turbine are turning at a steady rate.

The height, h metres, of the tip of one of the blades above the ground at time, t seconds, is given by the formula

$$h = 36\sin(1\cdot5t) - 15\cos(1\cdot5t) + 65.$$

Express $36\sin(1\cdot5t) - 15\cos(1\cdot5t)$ in the form

$$k\sin(1\cdot5t - a), \text{ where } k > 0 \text{ and } 0 < a < \frac{\pi}{2},$$

and hence find the **two** values of t for which the tip of this blade is at a height of 100 metres above the ground during the first turn. **8**

$k\sin(1.5t - a) = k\sin 1.5t\cos a - k\cos 1.5t\sin a$

$= k\cos a.\sin 1.5t - k\sin a.\cos 1.5t$

$\rightarrow 36.\sin 1.5t - 15.\cos 1.5t$

$k\cos a = 36$

$k\sin a = 15$

$k = \sqrt{36^2 + 15^2}$

$k = \sqrt{1521}$

$k = 39$

$\tan a = \frac{k\sin a}{k\cos a}$

$\tan a = \frac{15}{36} = \frac{5}{12}$

$ra = \tan^{-1}(5/12)$

$ra = 0.39479$ radians.

$a = 0.395$ (3dp)

↳ $36\sin(1.5t) - 15\cos(1.5t) = 39\sin(1.5t - 0.395)$

9) $h = 36\sin(1.5t) - 15\cos(1.5t) + 65$

$h = 39\sin(1.5t - 0.395) + 65$

At $h = 100$

$39\sin(1.5t - 0.395) + 65 = 100$ ✓

$39\sin(1.5t - 0.395) = 35$

$\sin(1.5t - 0.395) = \frac{35}{39}$ ✓

S ✓	A ✓
T	C

$ra = \sin^{-1}(35/39) = 1.1139...$

↳ $1.5t - 0.395 = 1.1139, \pi - 1.1139..$

$1.5t - 0.395 = 1.1139, 2.0276...$

$1.5t = 1.5089.., 2.4226..$ ✓

$t = 1.0059.., 1.6150...$

$t = 1.006, 1.615$ seconds. ✓