H

National
Qualifications
2015

X747/76/11

Mathematics
Paper 1
(Non-Calculator)

WEDNESDAY, 20 MAY
9:00 AM – 10:10 AM

60 marks

2015 PAPER 1 – WORKED SOLUTIONS

HWallace

1. Vectors $\mathbf{u} = 8\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{v} = -3\mathbf{i} + t\mathbf{j} - 6\mathbf{k}$ are perpendicular.

Determine the value of t. **2**

$\underset{\sim}{u} = \begin{pmatrix} 8 \\ 2 \\ -1 \end{pmatrix} \quad \underset{\sim}{v} = \begin{pmatrix} -3 \\ t \\ -6 \end{pmatrix}$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$ for perpendicular vectors

$8(-3) + 2(t) + (-1)(-6) = 0$ ✓

$-24 + 2t + 6 = 0$

$2t - 18 = 0$

$2t = 18$

$t = 9$ ✓

2. Find the equation of the tangent to the curve $y = 2x^3 + 3$ at the point where $x = -2$. **4**

Gradient $x = -2$

$y = 2x^3 + 3$

$\frac{dy}{dx} = 6x^2$ ✓

$m = 6(-2)^2$

$m = 6(4)$

$m = 24$ ✓

Point of Tangency

$y = 2(-2)^3 + 3$

$y = 2(-8) + 3$

$y = -16 + 3$

$y = -13$ ✓

P.O.T. $(-2, -13)$

Equation

$m = 24$

$a = -2$

$b = -13$

$y - b = m(x - a)$

$y + 13 = 24(x + 2)$

$y + 13 = 24x + 48$

$y = 24x + 35$ ✓

3. Show that $(x + 3)$ is a factor of $x^3 - 3x^2 - 10x + 24$ and hence factorise $x^3 - 3x^2 - 10x + 24$ fully. **4**

-3	1	-3	-10	24
	$\downarrow$	-3	18	-24
	1	-6	8	$\boxed{0}$

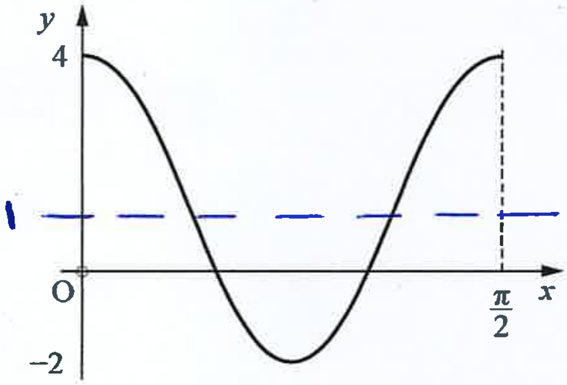
Since remainder is zero, then $x = -3$ is a root and $(x+3)$ is a factor!

$\hookrightarrow$ $x^3 - 3x^2 - 10x + 24$

$(x+3)(x^2 - 6x + 8)$

$(x+3)(x-2)(x-4)$

4. The diagram shows part of the graph of the function $y = p\cos qx + r$.



Amplitude $= \dfrac{4-(-2)}{2} = 3$

Period $= \dfrac{\pi}{2}$ or $90°$.

$\therefore$ 4 waves in 2π.

Write down the values of p, q and r. **3**

$y = 3\cos 4x + 1$

$p = 3$ $\quad$ $q = 4$ $\quad$ $r = 1$

5. A function g is defined on $\mathbb{R}$, the set of real numbers, by $g(x) = 6 - 2x$.

(a) Determine an expression for $g^{-1}(x)$. **2**

(b) Write down an expression for $g(g^{-1}(x))$. **1**

a) let $y = 6 - 2x$

$2x = 6 - y$

$x = \frac{6-y}{2}$ ✓

$\therefore g^{-1}(x) = \frac{6-x}{2}$ ✓

b) $g(g^{-1}(x)) = x$ ✓

6. Evaluate $\log_6 12 + \frac{1}{3}\log_6 27$. **3**

$= \log_6 12 + \log_6 27^{1/3}$ ✓

$27^{1/3} = \sqrt[3]{27} = 3$

$= \log_6 12 + \log_6 3$

$= \log_6 (12 \times 3)$ ✓

$= \log_6 36$

$= \log_6 6^2$

$= 2\log_6 6$

$= 2$ ✓

7. A function f is defined on a suitable domain by $f(x) = \sqrt{x}\left(3x - \dfrac{2}{x\sqrt{x}}\right)$.

Find $f'(4)$. Prepare! **4**

$$f(x) = x^{1/2}\left(3x^1 - \frac{2}{x(x^{1/2})}\right)$$

$$f(x) = 3x^{3/2} - \frac{2x^{1/2}}{x(x^{1/2})}$$

$$f(x) = 3x^{3/2} - \frac{2}{x}$$

$$f(x) = 3x^{3/2} - 2x^{-1} ✓$$

$$f'(x) = \frac{9}{2}x^{1/2} ✓ + 2x^{-2} ✓$$

$$f'(x) = \frac{9\sqrt{x}}{2} + \frac{2}{x^2}$$

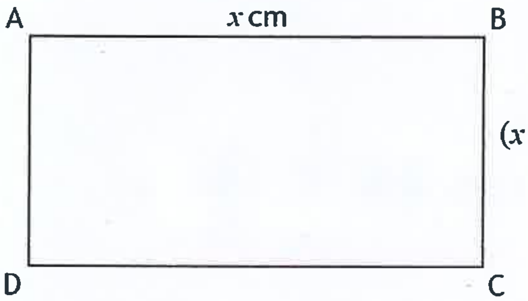
$$f'(4) = \frac{9\sqrt{4}}{2} + \frac{2}{4^2}$$

$$= \frac{9(2)}{2} + \frac{2}{16}$$

$$= 9 + \frac{1}{8}$$

$$f'(4) = 9\tfrac{1}{8} ✓$$

8. ABCD is a rectangle with sides of lengths x centimetres and $(x-2)$ centimetres, as shown.



length: $x > 0$

breadth: $x - 2 > 0$

$x > 2$

If the area of ABCD is less than $15\,\text{cm}^2$, determine the range of possible values of x. **4**

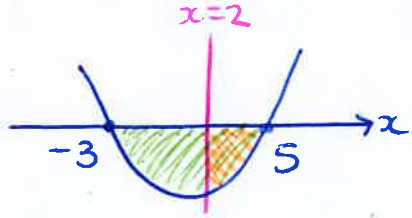
$$x(x-2) < 15 ✓$$

$$x^2 - 2x - 15 < 0 ✓$$

consider roots $x^2 - 2x - 15 = 0$

$(x+3)(x-5) ✓ = 0$

$x = -3 \quad x = 5$



Since $x > 2$ for breadth

then $2 < x < 5$ ✓

9. A, B and C are points such that AB is parallel to the line with equation $y+\sqrt{3}x=0$ and BC makes an angle of $150°$ with the positive direction of the x-axis.

Are the points A, B and C collinear? **3**

Parallel to AB

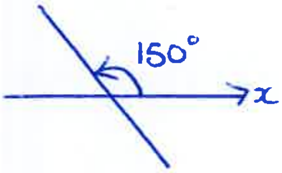
$y+\sqrt{3}x=0$

$y=-\sqrt{3}x$.

$\therefore\ m=-\sqrt{3}$.

$\hookrightarrow m_{AB}=-\sqrt{3}$ ✓

BC gives $\theta = 150°$.

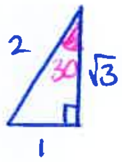


$m=\tan\theta$

$m_{BC}=\tan 150$

$m_{BC}=-\tan 30$

$m_{BC}=-\dfrac{1}{\sqrt{3}}$ ✓



Since $m_{AB}\neq m_{BC}$,
then points A, B and C are not collinear! ✓

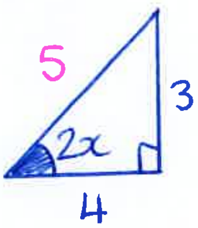
---

10. Given that $\tan 2x=\frac{3}{4}$, $0<x<\frac{\pi}{4}$, find the exact value of

(a) $\cos 2x$ **1**

(b) $\cos x$. **2**

a)



$\cos 2x = \dfrac{4}{5}$ ✓

b)

$\cos 2x = 2\cos^2 x - 1$

$\dfrac{4}{5} = 2\cos^2 x - 1$ ✓

$\dfrac{9}{5} = 2\cos^2 x$

$\dfrac{9}{10} = \cos^2 x$

$\cos x = \sqrt{\dfrac{9}{10}}$

$\cos x = \dfrac{3}{\sqrt{10}}$ ✓

11. T(−2, −5) lies on the circumference of the circle with equation

$$(x+8)^2 + (y+2)^2 = 45.$$

(a) Find the equation of the tangent to the circle passing through T. **4**

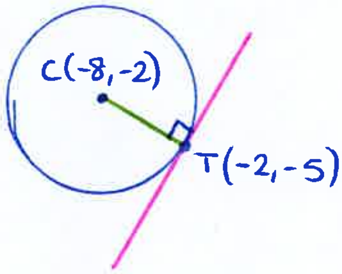
(b) This tangent is also a tangent to a parabola with equation $y = -2x^2 + px + 1 - p$, where $p > 3$.

Determine the value of p. **6**

a) $(x+8)^2 + (y+2)^2 = 45$

$(x-a)^2 + (y-b)^2 = r^2$

$a = -8 \qquad b = -2 \qquad C(-8, -2)$



$m_{radius} = \dfrac{-5-(-2)}{-2-(-8)} = \dfrac{-3}{6} = -\dfrac{1}{2}$

$m_{tangent} = 2$ since $m_1 \times m_2 = -1$ for $\perp$ lines.

Equation

$m = 2 \qquad y - b = m(x-a)$

$a = -2 \qquad y + 5 = 2(x+2)$

$b = -5 \qquad y + 5 = 2x + 4$

$y = 2x - 1.$

b) For point of intersection, let $y = y$

$2x - 1 = -2x^2 + px + 1 - p$

$2x^2 + 2x - px - 2 + p = 0$

$2x^2 + x(2-p) + (p-2) = 0$

$a = 2 \qquad b = (2-p) \qquad c = (p-2)$

11b) For tangency: $b^2 - 4ac = 0$

$$(2-p)^2 - 4(2)(p-2) = 0$$

$$4 - 4p + p^2 - 8p + 16 = 0$$

$$p^2 - 12p + 20 = 0$$

$$(p-10)(p-2) = 0$$

$p - 10 = 0$ or $\cancel{p - 2 = 0}$

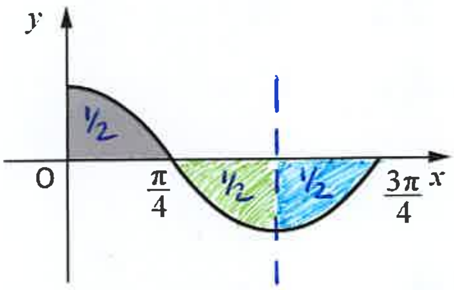
$p = 10$ $\qquad$ $\cancel{p = 2}$

no solution as $p > 3$.

↳ For tangency, $p = 10$.

12. The diagram shows part of the graph of $y = a\cos bx$.

The shaded area is $\frac{1}{2}$ unit^2.



POSITIVE VALUE

Area above $= \frac{1}{2}$ unit^2

Area below $= 1$ unit^2

NEGATIVE VALUE.

What is the value of $\int_0^{\frac{3\pi}{4}} (a\cos bx)\,dx$? **2**

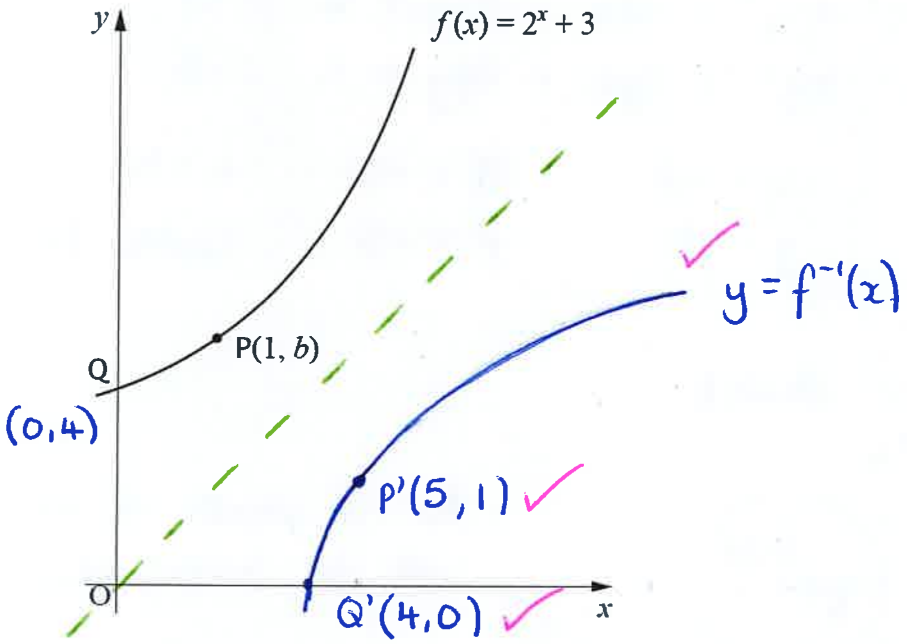
$$\int_0^{\frac{\pi}{4}} a\cos bx \, dx = \frac{1}{2}$$

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} a\cos bx \, dx = -1$$

↳ $\int_0^{\frac{3\pi}{4}} a\cos bx \, dx = -\frac{1}{2}$

13. The function $f(x) = 2^x + 3$ is defined on $\mathbb{R}$, the set of real numbers.

The graph with equation $y = f(x)$ passes through the point P$(1, b)$ and cuts the y-axis at Q as shown in the diagram.



(a) What is the value of b? **1**

(b) (i) Copy the above diagram.

On the same diagram, sketch the graph with equation $y = f^{-1}(x)$. **1**

(ii) Write down the coordinates of the images of P and Q. **3**

(c) R $(3,11)$ also lies on the graph with equation $y = f(x)$.

Find the coordinates of the image of R on the graph with equation $y = 4 - f(x+1)$. **2**

a) $y = 2^x + 3$ $\quad P(1, b)$ (x, y)

$b = 2^1 + 3$

$b = 5$

b) ii) For Q, $x = 0$

$y = 2^0 + 3 = 1 + 3$

$y = 4$

c) $y = 4 - f(x+1)$

$y = -f(x+1) + 4$

reflect on x-axis — left 1 — up 4.

(3, 11) (3, −11) (2, −11) (2, −7)

R(3, 11)

transforms to (2, −7)

14. The circle with equation $x^2 + y^2 - 12x - 10y + k = 0$ meets the coordinate axes at exactly three points.

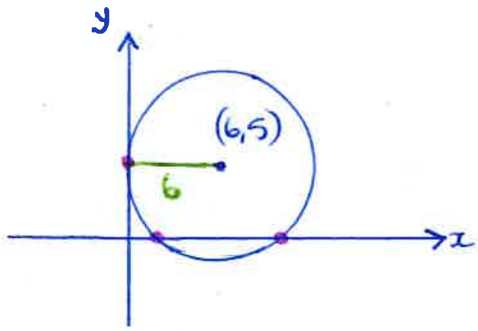
What is the value of k? **2**

$$x^2 + y^2 - 12x - 10y + k = 0$$
$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$2g = -12$ $\quad$ $2f = -10$ $\quad$ $c = k$.

$g = -6$ $\quad$ $f = -5$ $\quad$ Centre $(6, 5)$

Consider sketch.



For 3 points of intersection with the axes, then $r = 6$.

$$r = \sqrt{g^2 + f^2 - c}$$
$$6 = \sqrt{(-6)^2 + (-5)^2 - k}$$
$$36 = 36 + 25 - k$$
$$0 = 25 - k$$
$$k = 25.$$

15. The rate of change of the temperature, T °C of a mug of coffee is given by

$$\frac{dT}{dt} = \frac{1}{25}t - k \text{ , } 0 \leq t \leq 50$$

- t is the elapsed time, in minutes, after the coffee is poured into the mug
- k is a constant
- initially, the temperature of the coffee is 100 °C
- 10 minutes later the temperature has fallen to 82 °C.

Express T in terms of t. **6**

$$T = \int \frac{dT}{dt} \cdot dt$$

$$T = \int \frac{1}{25}t - k \;.\; dt$$

$$T = \frac{1}{25} \times \frac{t^2}{2} - kt + c$$

$$T = \frac{1}{50}t^2 - kt + c$$

At $t = 0$, then $T = 100$

$$100 = \frac{1}{50}(0)^2 - k(0) + c$$

$$100 = c$$

At $t = 10$, then $T = 82$

$$82 = \frac{1}{50}(10)^2 - k(10) + 100$$

$$82 = 2 - 10k + 100$$

$$80 = 100 - 10k$$

$$10k = 20$$

$$k = 2$$

$\hookrightarrow$ $T = \frac{1}{50}t^2 - 2t + 100$