HIGHER MATHS

Circle

Notes with Examples

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Equation of a Circle with Centre (a, b)

The equation of a circle with centre (a, b) and radius r, can be given by

$$(x - a)^2 + (y - b)^2 = r^2$$

To find out if a point lies inside, on or outside a circle you must substitute the x and y values into the equation.

If $(x - a)^2 + (y - b)^2 < r^2$ the point lies **inside** the circle.
If $(x - a)^2 + (y - b)^2 = r^2$ the point lies **on** the circle.
If $(x - a)^2 + (y - b)^2 > r^2$ the point lies **outside** the circle.

Examples

C-01 Write down the equation of the circle with given centre and radius

(a) $(2, 1);\ 7$ (b) $(-4, 2);\ 3$ (c) $(5, -4);\ diameter = 3$

(a) $(x-2)^2 + (y-1)^2 = 49$ (b) $(x+4)^2 + (y-2)^2 = 9$ (c) $(x-5)^2 + (y+4)^2 = \left(\frac{3}{2}\right)^2$

$(x-5)^2 + (y+4)^2 = \frac{9}{4}$

C-02 State the centre and radius of each of these circles

(a) $(x + 1)^2 + (y - 12)^2 = 40$

(b) $(x - 3)^2 + (y - 4)^2 = 225$

(a) CENTRE $(-1, 12)$ RADIUS $= \sqrt{40}$

$= 2\sqrt{10}$

(b) CENTRE $(3, 4)$ RADIUS $= \sqrt{225}$

$= 15$

C-03 Find the equation of the circle that passes through the point $(2,5)$ and has a centre of$(-1,2)$.

CENTRE $(-1, 2)$

RADIUS $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(2-(-1))^2 + (5-2)^2}$

$= \sqrt{9 + 9}$

$= \sqrt{18}$

CIRCLE

$\boxed{(x+1)^2 + (y-2)^2 = 18}$

C-04 Given $(x - 3)^2 + (y + 2)^2 = 100$ is a circle, does A(9,-6) lie inside, on or outside the circle?

SUBSTITUTE $x = 9$ $y = -6$

$(9 - 3)^2 + (-6 + 2)^2$

$= (6)^2 + (-4)^2$

$= 36 + 16$

$= 52$

$\boxed{52 < 100 \text{ SO A IS INSIDE CIRCLE}}$

General Equation of a Circle

The general equation of a circle is given by

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

with a centre of $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$

This is given in a formulae sheet!

Note: For a circle to exist then $g^2 + f^2 - c > 0$

To find out if a point lies inside, on or outside a circle you must substitute the x and y values into the equation.

If $x^2 + y^2 + 2gx + 2fy + c < 0$ the point lies **inside** the circle.
If $x^2 + y^2 + 2gx + 2fy + c = 0$ the point lies **on** the circle.
If $x^2 + y^2 + 2gx + 2fy + c > 0$ the point lies **outside** the circle.

Examples

C-05 Do these equations represent circles?

(a) $x^2 + y^2 - 6x + 2y - 2 = 0$

(b) $x^2 + y^2 + 2x - 2y + 9 = 0$

(a) $g^2 + f^2 - c$

$= (-3)^2 + (1)^2 - (-2)$

$= 12$

$\boxed{12 > 0 \text{ SO THIS IS A CIRCLE}}$

(b) $g^2 + f^2 - c$

$= (1)^2 + (-1)^2 - 9$

$= -7$

$\boxed{-7 < 0 \text{ SO THIS IS NOT A CIRCLE}}$

C-06 State the centre and radius of each of these circles

(a) $x^2 + y^2 - 8x - 10y + 3 = 0$

(b) $x^2 + y^2 - 4x + 6y - 5 = 0$

(a) $2g = -8$ $\quad 2f = -10$ $\quad c = 3$

$g = -4$ $\quad f = -5$

CENTRE $(4, 5)$ $\quad$ RADIUS $= \sqrt{(-4)^2 + (-5)^2 - 3}$

$= \sqrt{38}$

(b) $2g = -4$ $\quad 2f = 6$ $\quad c = -5$

$g = -2$ $\quad f = 3$

CENTRE $(2, -3)$ $\quad$ RADIUS $= \sqrt{(-2)^2 + (3)^2 - (-5)}$

$= \sqrt{18}$

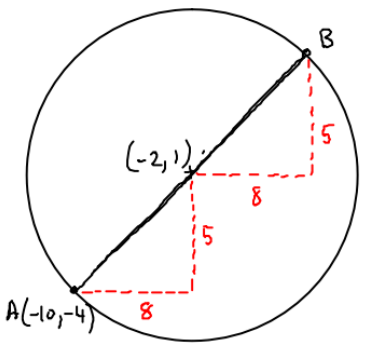
$= 3\sqrt{2}$

C-07 A circle $x^2 + y^2 + 4x - 2y - 84 = 0$ has a diameter AB. If A is the point $(-10, -4)$, find the coordinates of B.

CENTRE $= (-2, 1)$

$B = (-2 + 8, 1 + 5)$

$= (6, 6)$



C-08 Given $x^2 + y^2 - 8x - 10y - 8 = 0$ is a circle, does A$(1, -2)$ lie inside, on or outside the circle?

$x^2 + y^2 - 8x - 10y - 8$

$= (1)^2 + (-2)^2 - 8(1) - 10(-2) - 8$

$= 1 + 4 - 8 + 20 - 8$

$= 9$

$9 > 0$ SO POINT A LIES OUTSIDE THE CIRCLE

Circle in Context

Examples

C-09 The circle $x^2 + y^2 + 2x - 14y - 15 = 0$ cuts the x-axis at points A and B, and the y-axis at C and D. Calculate the length of AB and CD.

CUTS X AXIS WHEN $y = 0$

$$x^2 + 2x - 15 = 0$$
$$(x+5)(x-3) = 0$$
$$x = -5 \qquad x = 3$$

$\boxed{\text{LENGTH AB} = 8}$

CUTS Y AXIS WHEN $x = 0$

$$y^2 - 14y - 15 = 0$$
$$(y-15)(y+1) = 0$$
$$y = 15 \qquad y = -1$$

$\boxed{\text{LENGTH CD} = 16}$

C-10 Circle C_1 and C_2 are concentric. The larger circle has the equation $x^2 + y^2 - 4x - 6y - 68 = 0$. The radius of the smaller circle is half that of the larger circle. State the equation of the smaller circle.

LARGE CIRCLE

$$2g = -4 \qquad 2f = -6 \qquad c = -68$$
$$g = -2 \qquad f = -3$$

CENTRE $(2, 3)$

$$\text{RADIUS} = \sqrt{(-2)^2 + (-3)^2 - (-68)}$$
$$= \sqrt{81}$$
$$= 9$$

SMALL CIRCLE

CENTRE $(2, 3)$

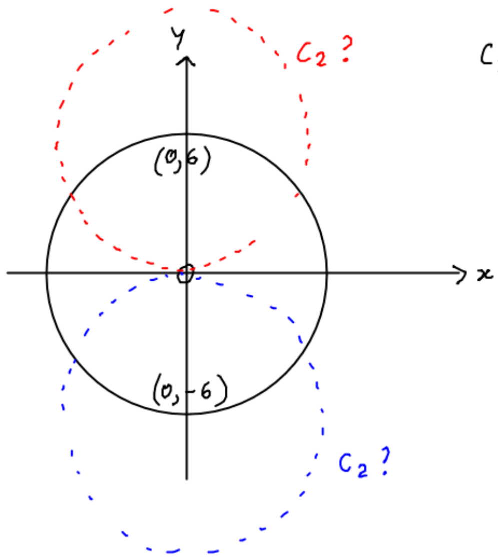
CONCENTRIC CIRCLES HAVE SAME CENTRES

$$\text{RADIUS} = \frac{1}{2} \times 9$$
$$= \frac{9}{2}$$

$$\boxed{\text{EQUATION: } (x-2)^2 + (y-3)^2 = \frac{81}{4}}$$

$$\left(\frac{9}{2}\right)^2 = \frac{81}{4}$$

C-11 Circle C_1 and C_2 are identical. C_1 has a centre $(0,0)$ and a radius of 6 units. Circle C_2 passes through the origin and its centre is on the y-axis. State both possible equations for circle C_2.



$C_2 \quad (x-0)^2 + (y-6)^2 = 36$

$$\boxed{\begin{array}{c} x^2 + (y-6)^2 = 36 \\ \text{OR} \\ x^2 + (y+6)^2 = 36 \end{array}}$$

Points of Intersection with a Straight Line

There are three possibilities for the point of intersection between a circle and a straight line:

* the circle and line intersect at two points
* the circle and line intersect at one point (tangent)
* the circle and line do not intersect

This is similar to quadratic roots: two distinct roots, one repeated root and no real roots.

To find points of intersection we substitute the equation of the straight line into the circle and solve.

Examples

C-12 Find the points of intersection of these circles and lines

(a) $x^2 + y^2 = 10;\ y = 3$

(b) $x^2 + y^2 + 2x - 2y - 11 = 0;\ 5y - x + 7 = 0$

(c) $x^2 + y^2 - 8x - 4y - 20 = 0;\ y = 3x + 10$

(d) $x^2 + y^2 - 8x - 10y - 8 = 0;\ y = -2x - 5$

(a)
$$x^2 + (3)^2 = 10$$
$$x^2 + 9 = 10$$
$$x^2 = 1$$
$$x = \pm 1$$

$\boxed{\text{POI } (1,3) \text{ AND } (-1,3)}$

(b)
$$5y - x + 7 = 0$$
$$5y + 7 = x$$

$$(5y+7)^2 + y^2 + 2(5y+7) - 2y - 11 = 0$$
$$25y^2 + 70y + 49 + y^2 + 10y + 14 - 2y - 11 = 0$$
$$26y^2 + 78y + 52 = 0$$
$$26(y^2 + 3y + 2) = 0$$
$$26(y+2)(y+1) = 0$$
$$y = -2 \quad y = -1$$

WHEN $y = -1$ $\quad x = 5(-1) + 7 = 2$

WHEN $y = -2$ $\quad x = 5(-2) + 7 = -3$

$\boxed{\text{POI } (-3,-2) \text{ AND } (2,-1)}$

(c) $x^2 + (3x+10)^2 - 8x - 4(3x+10) - 20 = 0$

$x^2 + 9x^2 + 60x + 100 - 8x - 12x - 40 - 20 = 0$

$10x^2 + 40x + 40 = 0$

$10(x^2 + 4x + 4) = 0$

$10(x+2)(x+2) = 0$

$x = -2$

WHEN $x = -2$ $\quad y = 3(-2) + 10$

$= 4$

$\boxed{\text{PoC } (-2, 4)}$

$y = 3x + 10$ IS A TANGENT.

(d) $x^2 + (-2x-5)^2 - 8x - 10(-2x-5) - 8 = 0$

$x^2 + 4x^2 + 20x + 25 - 8x + 20x + 50 - 8 = 0$

$5x^2 + 32x + 67 = 0$

$b^2 - 4ac$

$= 32^2 - 4 \times 5 \times 67$

$= -316$

$b^2 - 4ac < 0$

SO NO POINTS OF INTERSECTION

Tangents to a Circle

A tangent is a straight line that touches a circle at one point. At this point, the radius of the circle and the straight line are at right angles.

To find the equation of a tangent follow these steps:

* calculate the gradient of the radius (centre to POC)
* calculate $m_{\perp}$
* substitute POC and $m_{\perp}$ into $y - b = m(x - a)$

Examples

C-13 Find the equation of the tangent to these circles at the given points:

(a) $x^2 + y^2 - 4x - 2y - 3 = 0$; $(4,3)$

(b) $(x - 2)^2 + (y - 6)^2 = 9$; $(2,3)$

(a) CENTRE $(2,1)$

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{3-1}{4-2}$

$= 1$

$m_{\perp} = -1$

$y - b = m(x - a)$

$y - 3 = -1(x - 4)$

$y - 3 = -x + 4$

$\boxed{y = -x + 7}$

(b) CENTRE $(2,6)$

$\boxed{y = 3}$

$m =$ undefined

$m_{\perp} = 0$

C-14 Show that the line $x + 3y - 11 = 0$ is a tangent to the circle $x^2 + y^2 + 2x + 12y - 53 = 0$ and find the point of contact.

$x = -3y + 11$

$(-3y+11)^2 + y^2 + 2(-3y+11) + 12y - 53 = 0$

$9y^2 - 66y + 121 + y^2 - 6y + 22 + 12y - 53 = 0$

$10y^2 - 60y + 90 = 0$

$10(y^2 - 6y + 9) = 0$

$10(y-3)(y-3) = 0$

REPEATED ROOT SO TANGENT

$y = 3$

$x + 3(3) - 11 = 0$

$x + 9 - 11 = 0$

$x - 2 = 0$

$x = 2$

POC $(2,3)$

C-15 Find the point of intersection, P, of the tangents to the circle $x^2 + y^2 - 8x - 4y + 10 = 0$ at the points $A(1,1)$ and $B(5,5)$ and prove that $AP = BP$.

CENTRE $(4,2)$

$m_{AC} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-2}{1-4} = \frac{1}{3}$

$m_{BC} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-2}{5-4} = 3$

$y - 1 = \frac{1}{3}(x-1)$

$y - 1 = \frac{1}{3}x - \frac{1}{3}$

$y = \frac{1}{3}x + \frac{2}{3}$

$y - 5 = 3(x-5)$

$y - 5 = 3x - 15$

$y = 3x - 10$

POINT P

$3x - 10 = \frac{1}{3}x + \frac{2}{3}$

$9x - 30 = x + 2$

$8x = 32$

$x = 4$

$y = 3(4) - 10$
$= 2$

P $(4,2)$

$AP = \sqrt{(1-4)^2 + (1-2)^2}$
$= \sqrt{10}$

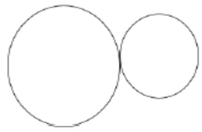
$BP = \sqrt{(5-4)^2 + (5-2)^2}$
$= \sqrt{10}$

$AP = BP = \sqrt{10}$

Intersecting Circles

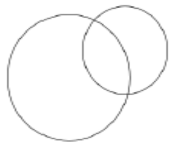
To determine whether or not two circles intersect we compare the distance between the two centres, d, and both radii. There are five possibilities:

* The circles meet externally at one point. The distance between centres is equal to the sum of the radii.



$$d = r_1 + r_2$$

* The circles meet at two points. The distance between centres is less than the sum of the radii.

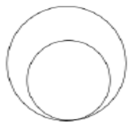


$$d < r_1 + r_2$$

* The circles don't meet. The distance between centres is greater than the sum of the radii.

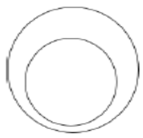
$$d > r_1 + r_2$$

* The circles meet at one point and one circle is inside the other. The distance between centres is equal to the difference of the radii.



$$d = r_1 - r_2$$

* The circles don't touch and one circle is inside the other. The distance between centres is less than the difference of the radii.



$$d < r_1 - r_2$$

Examples

C-16 Determine how, if at all, these circles intersect:

(a) $x^2 + y^2 + 16x + 8y - 320 = 0$; $x^2 + y^2 + 8x + 2y - 47 = 0$

(b) $x^2 + y^2 - 8x + 6y - 11 = 0$; $x^2 + y^2 + 4x - 10y + 4 = 0$

(c) $x^2 + y^2 - 2x + 4y - 14 = 0$; $x^2 + y^2 + 4x - 10y + 27 = 0$

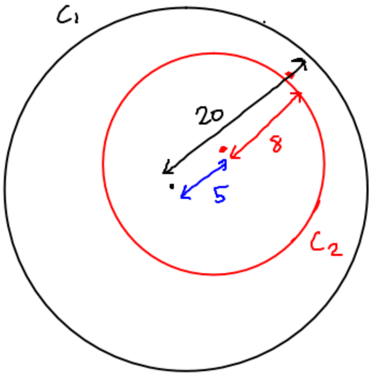
(a) $C_1 \ (-8, -4)$ $\qquad$ $C_2 = (-4, -1)$

$$r_1 = \sqrt{8^2 + 4^2 - (-320)}$$
$$= \sqrt{400}$$
$$= 20$$

$$r_2 = \sqrt{(-4)^2 + (-1)^2 - (-47)}$$
$$= \sqrt{64}$$
$$= 8$$

listance between centres $= \sqrt{((-8)+(-4))^2 + ((-4)+(-1))^2}$

$$= \sqrt{16 + 9}$$
$$= 5$$



CIRCLE C_2 IS INSIDE CIRCLE C_1 AND THERE ARE NO POINTS OF CONTACT

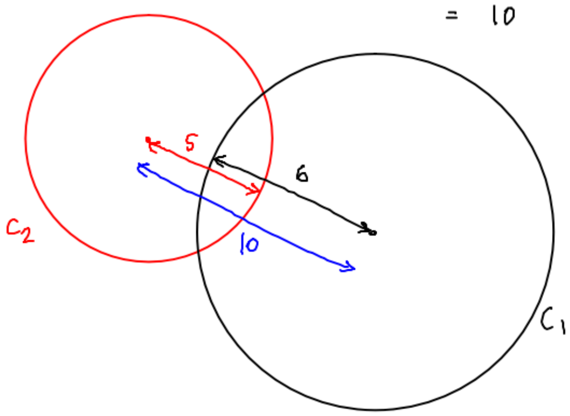
(b) $C_1 = (4, -3)$ $\qquad$ $C_2 = (-2, 5)$

$$r_1 = \sqrt{(-4)^2 + (3)^2 - (-11)}$$
$$= \sqrt{36}$$
$$= 6$$

$$r_2 = \sqrt{(2)^2 + (-5)^2 - 4}$$
$$= \sqrt{25}$$
$$= 5$$

distance between centres $= \sqrt{(4 - (-2))^2 + ((-3) - 5)^2}$

$$= \sqrt{36 + 64}$$
$$= 10$$



C_1 AND C_2 ARE EXTERNAL AND HAVE TWO POINTS OF INTERSECTION

(c) $C_1 = (1, -2)$ $\qquad C_2 = (-2, 5)$

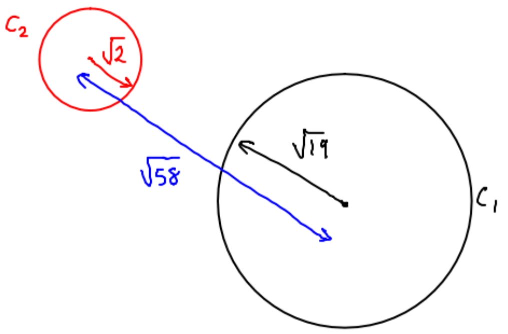
$r_1 = \sqrt{(-1)^2 + (2)^2 - (-14)}$ $\qquad r_2 = \sqrt{(2)^2 + (-5)^2 - 27}$

$= \sqrt{19}$ $(= 4.35)$ $\qquad = \sqrt{2}$ $(= 1.41)$

distance between centres $= \sqrt{(1-(-2))^2 + ((-2)-5)^2}$

$= \sqrt{9 + 49}$

$= \sqrt{58}$ $(= 7.61)$



$\boxed{\text{C}_1 \text{ AND } \text{C}_2 \text{ ARE EXTERNAL AND HAVE } \underline{\text{NO}} \text{ POINTS OF INTERSECTION}}$

Summary

the Circle

EQUATION OF A CIRCLE :

$$\boxed{(x-a)^2 + (y-b)^2 = r^2}$$

where centre (a,b) radius $= r$

OR

$$\boxed{x^2 + y^2 + 2gx + 2fy + c = 0}$$

where centre $(-g,-f)$ and radius $= \sqrt{g^2 + f^2 - c}$

BOTH ARE GIVEN IN FORMULAE SHEET !

POINTS OF INTERSECTION / TANGENCY :

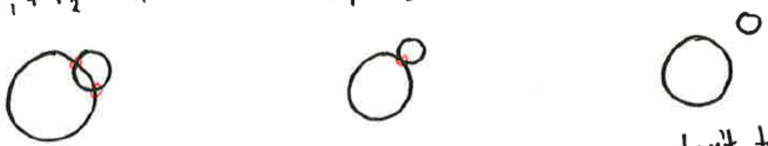
Substitute equation of line into the circle, use discriminant of resultant quadratic

$b^2 - 4ac > 0$	$b^2 - 4ac = 0$	$b^2 - 4ac < 0$
two points	tangent	no points

similar to parabolas !

DO TWO CIRCLES TOUCH ?

to find out :
1. calculate r_1 and r_2
2. calculate distance between centres (d)
3. Compare

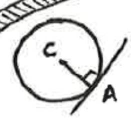


$r_1 + r_2 > d$	$r_1 + r_2 = d$	$r_1 + r_2 < d$
touch at two points	touch at one point	don't touch at all

★ two further cases occur when one circle is inside the other ★

dont touch	touch at one point
$r_1 - r_2 > r$	$r_1 - r_2 = d$

CONCENTRIC : CIRCLES WITH THE SAME CENTRE



EQUATION OF A TANGENT :

1. m_{AC}
2. $m_1 \times m_2 = -1$ for $m_\perp$
3. Use point A & $m_\perp$ for equation