HIGHER MATHS

2017 Exam

Question Paper with Solutions

Mr Miscandlon
gw13miscandlondavid@glow.sch.uk

2017 Non Calculator Paper

1. Functions f and g are defined on suitable domains by $f(x)=5x$ and $g(x)=2\cos x$.

(a) Evaluate $f(g(0))$. **1**

(b) Find an expression for $g(f(x))$. **2**

(a) $f(g(x)) = f(2\cos x)$

$= 5(2\cos x)$

$= 10\cos x$

$f(g(0)) = 10\cos 0$

$= 10 \times 1$

$\boxed{f(g(0)) = 10}$

(b) $g(f(x)) = g(5x)$

$\boxed{g(f(x)) = 2\cos 5x}$

2. The point P $(-2, 1)$ lies on the circle $x^2+y^2-8x-6y-15=0$.
Find the equation of the tangent to the circle at P. **4**

CENTRE $(4, 3)$

$m_{RADIUS} = \dfrac{y_2-y_1}{x_2-x_1}$

$= \dfrac{3-1}{4-(-2)}$

$= \dfrac{2}{6}$

$= \dfrac{1}{3}$

$m_1 \times m_2 = -1$

$m_\perp = -3$

$y-b = m(x-a)$

$y-1 = -3(x+2)$

$y-1 = -3x-6$

$\boxed{y = -3x-5}$

3. Given $y=(4x-1)^{12}$, find $\dfrac{dy}{dx}$. **2**

$y = (4x-1)^{12}$

$\dfrac{dy}{dx} = 12(4x-1)^{11} \times 4$

$\boxed{\dfrac{dy}{dx} = 48(4x-1)^{11}}$

4. Find the value of k for which the equation $x^2 + 4x + (k-5) = 0$ has equal roots. **3**

FOR EQUAL ROOTS $b^2 - 4ac = 0$

$(4)^2 - (4 \times 1 \times (k-5)) = 0$

$16 - (4(k-5)) = 0$

$16 - 4k + 20 = 0$

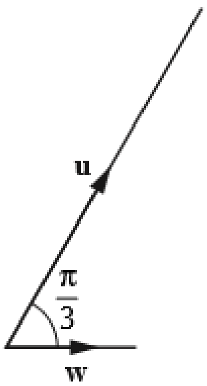
$-4k = -36$

$\boxed{k = 9}$

5. Vectors $\mathbf{u}$ and $\mathbf{v}$ are $\begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -8 \\ 6 \end{pmatrix}$ respectively.

(a) Evaluate $\mathbf{u}.\mathbf{v}$. **1**

(b)



Vector $\mathbf{w}$ makes an angle of $\frac{\pi}{3}$ with $\mathbf{u}$ and $|\mathbf{w}| = \sqrt{3}$.

Calculate $\mathbf{u}.\mathbf{w}$. **3**

(a) $\underline{u}.\underline{v}$

$= 5(3) + 1(-8) + (-1)(6)$

$= 15 - 8 - 6$

$\boxed{\underline{u}.\underline{v} = 1}$

(b) $|\underline{u}| = \sqrt{(5)^2 + (1)^2 + (-1)^2}$

$= \sqrt{27}$

$\underline{u}.\underline{w} = |\underline{u}||\underline{w}| \cos\theta$

$= \sqrt{27} \times \sqrt{3} \times \cos\frac{\pi}{3}$

$= \sqrt{81} \times \frac{1}{2}$

$\boxed{\underline{u}.\underline{w} = \frac{9}{2}}$

6. A function, h, is defined by $h(x) = x^3 + 7$, where $x \in \mathbb{R}$.

Determine an expression for $h^{-1}(x)$. **3**

$$y = x^3 + 7$$
$$x^3 + 7 = y$$
$$x^3 = y - 7$$
$$x = \sqrt[3]{y - 7}$$
$$\boxed{h^{-1}(x) = \sqrt[3]{x - 7}}$$

7. A $(-3, 5)$, B $(7, 9)$ and C $(2, 11)$ are the vertices of a triangle.

Find the equation of the median through C. **3**

$$Q = \text{MID}_{AB} = \left(\frac{(-3)+7}{2}, \frac{5+9}{2} \right)$$
$$= (2, 7)$$

$$m_{CQ} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{7 - 11}{2 - 2}$$
$$= \frac{-4}{0} = \text{VERTICAL LINE}$$

$$\boxed{x = 2}$$

8. Calculate the rate of change of $d(t) = \frac{1}{2t}$, $t \neq 0$, when $t = 5$. **3**

$$d(t) = \frac{1}{2t}$$
$$= \frac{1}{2} t^{-1}$$
$$d'(t) = -\frac{1}{2} t^{-2}$$
$$= -\frac{1}{2t^2}$$
$$d'(5) = \frac{-1}{2(5)^2}$$
$$\boxed{d'(5) = -\frac{1}{50}}$$

9. A sequence is generated by the recurrence relation $u_{n+1} = mu_n + 6$ where m is a constant.

(a) Given $u_1 = 28$ and $u_2 = 13$, find the value of m. **2**

(b) (i) Explain why this sequence approaches a limit as $n \to \infty$. **1**

(ii) Calculate this limit. **2**

(a) $u_{n+1} = mu_n + 6$

$13 = 28m + 6$

$7 = 28m$

$m = \frac{7}{28}$

$\boxed{m = \frac{1}{4}}$

(b)(i) LIMIT EXISTS AS $-1 < \frac{1}{4} < 1$

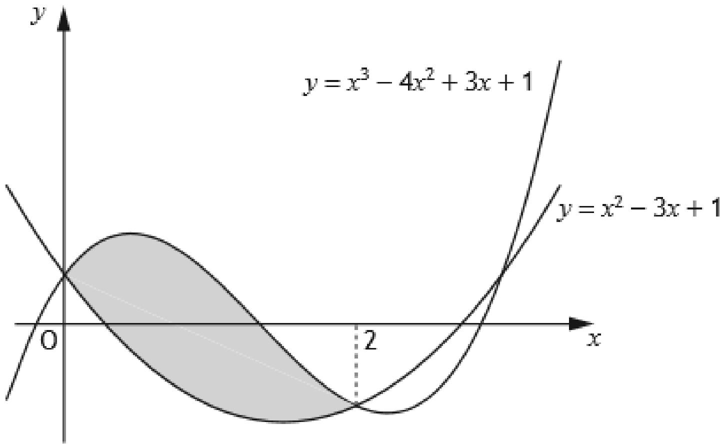
(ii) $L = \frac{1}{4}L + 6$

$\frac{3}{4}L = 6$

$3L = 24$

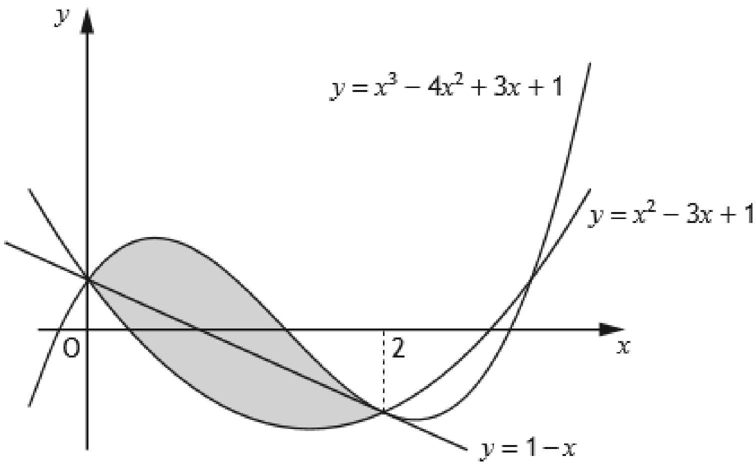
$\boxed{L = 8}$

10. Two curves with equations $y = x^3 - 4x^2 + 3x + 1$ and $y = x^2 - 3x + 1$ intersect as shown in the diagram.



(a) Calculate the shaded area. **5**

The line passing through the points of intersection of the curves has equation $y = 1 - x$.



(b) Determine the fraction of the shaded area which lies below the line $y = 1 - x$. **4**

(a) AREA $= \int_0^2$ UPPER − LOWER

$= \int_0^2 (x^3 - 4x^2 + 3x + 1) - (x^2 - 3x + 1)\,dx$

$= \int_0^2 x^3 - 5x^2 + 6x \; dx$

$= \left[\frac{x^4}{4} - \frac{5x^3}{3} + 3x^2\right]_0^2$

$= \left(\frac{(2)^4}{4} - \frac{5(2)^3}{3} + 3(2)^2\right) - (0)$

$= 4 - \frac{40}{3} + 12$

$\boxed{\text{AREA } = \frac{8}{3} \text{ units}^2}$

(b) AREA $= \int_0^2$ UPPER − LOWER

$= \int_0^2 (1 - x) - (x^2 - 3x + 1)\,dx$

$= \int_0^2 -x^2 + 2x \; dx$

$= \left[-\frac{x^3}{3} + x^2\right]_0^2$

$= \left(-\frac{(2)^3}{3} + (2)^2\right) - (0)$

$= -\frac{8}{3} + 4$

$= \frac{4}{3}$ units^2 $\quad \frac{4}{3}$ IS $\frac{1}{2}$ OF $\frac{8}{3}$

SO HALF SHADED AREA IS BELOW THE LINE

11. A and B are the points $(-7, 2)$ and $(5, a)$.

AB is parallel to the line with equation $3y - 2x = 4$.

Determine the value of a. **3**

$$3y - 2x = 4$$
$$3y = 2x + 4$$
$$y = \frac{2}{3}x + \frac{4}{3}$$
$$\boxed{m = \frac{2}{3}}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{a - 2}{5 - (-7)}$$
$$= \frac{a - 2}{12}$$

$$\frac{a-2}{12} = \frac{2}{3}$$
$$a - 2 = 12 \times \frac{2}{3}$$
$$a - 2 = 8$$
$$\boxed{a = 10}$$

12. Given that $\log_a 36 - \log_a 4 = \frac{1}{2}$, find the value of a. **3**

$$\log_a 36 - \log_a 4 = \frac{1}{2}$$
$$\log_a\left(\frac{36}{4}\right) = \frac{1}{2}$$
$$\log_a 9 = \frac{1}{2}$$
$$a^{1/2} = 9$$
$$\boxed{a = 81}$$

13. Find $\int \frac{1}{(5-4x)^{\frac{1}{2}}}dx,\ x < \frac{5}{4}$. **4**

$$\int \frac{1}{(5-4x)^{1/2}}\,dx$$
$$= \int (5-4x)^{-1/2}\,dx$$
$$= \frac{(5-4x)^{1/2}}{1/2} \times \frac{1}{-4} + c$$
$$\boxed{= -\frac{(5-4x)^{1/2}}{2} + c}$$

14. (a) Express $\sqrt{3}\sin x° - \cos x°$ in the form $k\sin(x-a)°$, where $k > 0$ and $0 < a < 360$. **4**

(b) Hence, or otherwise, sketch the graph with equation $y = \sqrt{3}\sin x° - \cos x°$, $0 \le x \le 360$. **3**

Use the diagram provided in the answer booklet.

(a) $\underline{\sqrt{3}\sin x} - \underline{\underline{\cos x}} = k\sin(x-a)$

$= k\underline{\sin x\cos a} - k\underline{\underline{\cos x}}\sin a$

$-k\sin a = -1$

$k\sin a = 1$

$k\cos a = \sqrt{3}$

$\tan a = \frac{1}{\sqrt{3}}$

$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30°$

✓S	A ✓✓✓
✓T	C ✓

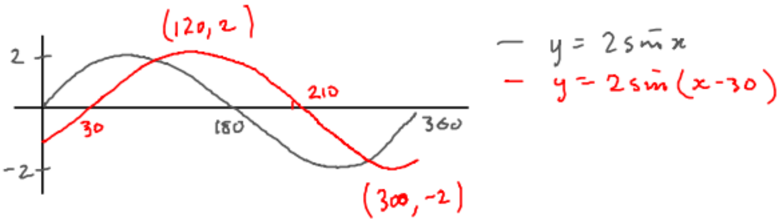
$a = 30°$

$k = \sqrt{(1)^2 + (\sqrt{3})^2}$

$= 2$

$$\boxed{\sqrt{3}\sin x - \cos x = 2\sin(x - 30°)}$$

(b)



15. A quadratic function, f, is defined on $\mathbb{R}$, the set of real numbers.

Diagram 1 shows part of the graph with equation $y = f(x)$.
The turning point is $(2, 3)$.

Diagram 2 shows part of the graph with equation $y = h(x)$.
The turning point is $(7, 6)$.

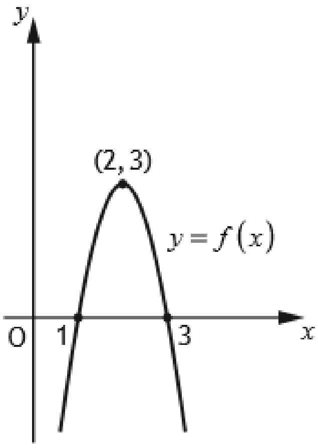


Diagram 1

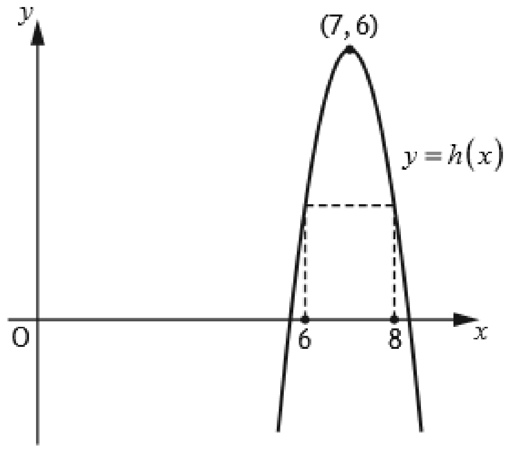


Diagram 2

(a) Given that $h(x) = f(x+a)+b$.

Write down the values of a and b. **2**

(b) It is known that $\int_1^3 f(x)\,dx = 4$.

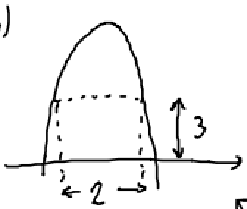
Determine the value of $\int_6^8 h(x)\,dx$. **1**

(c) Given $f'(1) = 6$, state the value of $h'(8)$. **1**

(a) $h(x) = f(x-5)+3$

$\boxed{a = -5 \quad b = 3}$

(b)



$\int_6^8 h(x)$

$= 4 + (3\times 2)$

$= 10$

$\boxed{\int_6^8 h(x) = 10}$

(c)

$f'(1) = 6$, $f'(3) = -6$

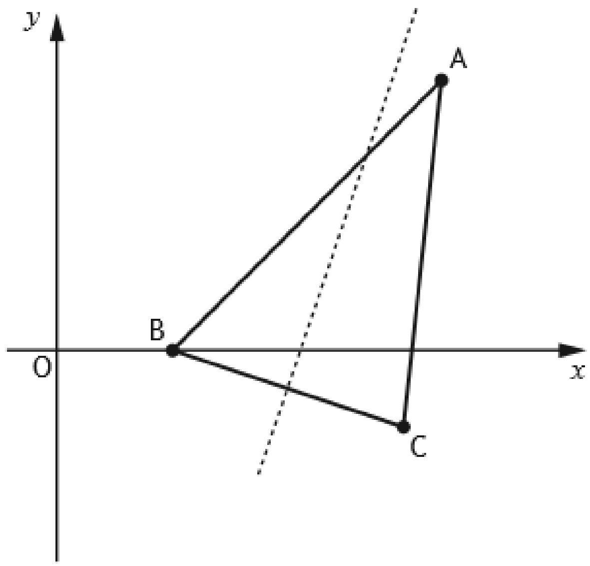
$\boxed{h'(8) = -6}$

2017 Calculator Paper

1. Triangle ABC is shown in the diagram below.

   The coordinates of B are (3,0) and the coordinates of C are (9,–2).

   The broken line is the perpendicular bisector of BC.



(a) Find the equation of the perpendicular bisector of BC. **4**

(b) The line AB makes an angle of 45° with the positive direction of the x-axis.

Find the equation of AB. **2**

(c) Find the coordinates of the point of intersection of AB and the perpendicular bisector of BC. **2**

(a) $Q = \text{mid}_{BC} = \left(\frac{3+9}{2}, \frac{0+(-2)}{2}\right)$

$\boxed{= (6,-1)}$

$m_{BC} = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{-2-0}{9-3}$

$= -\frac{2}{6}$

$= -\frac{1}{3}$

$m_1 \times m_2 = -1$

$\boxed{m_\perp = 3}$

$y - b = m(x-a)$

$y + 1 = 3(x-6)$

$\boxed{y = 3x - 19}$

(b) $m = \tan\theta$

$= \tan 45$

$\boxed{m = 1}$

$y - b = m(x-a)$

$y - 0 = 1(x-3)$

$\boxed{y = x - 3}$

(c) PoI

$3x - 19 = x - 3$

$3x - x = -3 + 19$

$2x = 16$

$x = 8$

$y = (8) - 3$

$= 5$

$\boxed{\text{PoI } (8,5)}$

2. (a) Show that $(x-1)$ is a factor of $f(x)=2x^3-5x^2+x+2$. **2**

(b) Hence, or otherwise, solve $f(x)=0$. **3**

(a)

$$\begin{array}{c|cccc} 1 & 2 & -5 & 1 & 2 \\ & & 2 & -3 & -2 \\ \hline & 2 & -3 & -2 & \boxed{0} \end{array}$$

$\boxed{\text{REM} = 0 \text{ SO } x-1 \text{ IS A FACTOR}}$

(b)

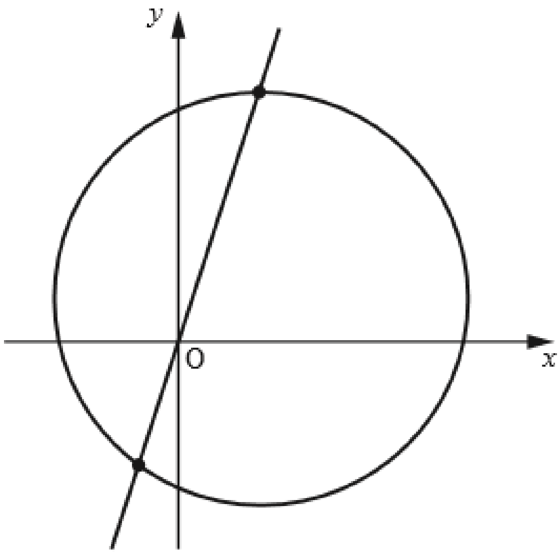
$2x^3-5x^2+x+2=0$

$(x-1)(2x^2-3x-2)=0$

$(x-1)(2x+1)(x-2)=0$

$\boxed{x=1 \quad x=-\frac{1}{2} \quad x=2}$

3. The line $y=3x$ intersects the circle with equation $(x-2)^2+(y-1)^2=25$.



Find the coordinates of the points of intersection. **5**

$(x-2)^2+(y-1)^2=25$

$(x-2)^2+(3x-1)^2=25$

$x^2-4x+4+9x^2-6x+1=25$

$10x^2-10x-20=0$

$10(x^2-x-2)=0$

$10(x-2)(x+1)=0$

$x=2 \qquad x=-1$

$y=3(2) \qquad y=3(-1)$

$=6 \qquad =-3$

$\boxed{\text{POI } (2,6) \text{ AND } (-1,-3)}$

4. (a) Express $3x^2+24x+50$ in the form $a(x+b)^2+c$. **3**

(b) Given that $f(x)=x^3+12x^2+50x-11$, find $f'(x)$. **2**

(c) Hence, or otherwise, explain why the curve with equation $y=f(x)$ is strictly increasing for all values of x. **2**

(a) $3x^2+24x+50$

$= 3\left[x^2+8x\right]+50$

$= 3\left[(x+4)^2-16\right]+50$

$= 3(x+4)^2-48+50$

$\boxed{= 3(x+4)^2+2}$

(b) $f(x) = x^3+12x^2+50x-11$

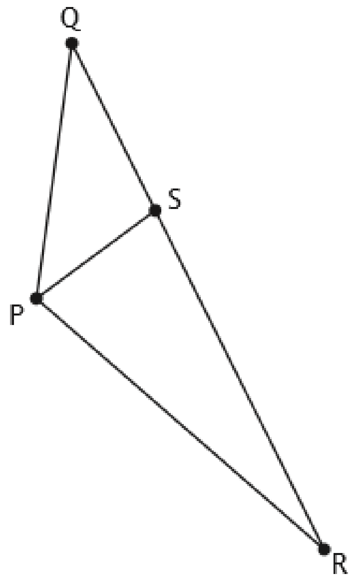
$f'(x) = 3x^2+24x+50$

(c) $f'(x) = 3x^2+24x+50$

$= 3(x+4)^2+2$

MIN VALUE OF $f'(x)=2$
SO FUNCTION ALWAYS
INCREASING

5. In the diagram, $\overrightarrow{PR} = 9\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{RQ} = -12\mathbf{i} - 9\mathbf{j} + 3\mathbf{k}$.



(a) Express $\overrightarrow{PQ}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. **2**

The point S divides QR in the ratio 1:2.

(b) Show that $\overrightarrow{PS} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$. **2**

(c) Hence, find the size of angle QPS. **5**

(a) $\overrightarrow{PQ} = \overrightarrow{PR} + \overrightarrow{RQ}$

$= \begin{pmatrix} 9 \\ 5 \\ 2 \end{pmatrix} + \begin{pmatrix} -12 \\ -9 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix}$

$\boxed{\overrightarrow{PQ} = -3i - 4j + 5k}$

(b) $\overrightarrow{PS} = \overrightarrow{PQ} + \frac{1}{3}\overrightarrow{QR}$

$= \begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 12 \\ 9 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$

$\boxed{\overrightarrow{PS} = i - j + 4k}$

(c) $\cos QRS = \dfrac{\overrightarrow{PQ} \cdot \overrightarrow{PS}}{|\overrightarrow{PQ}||\overrightarrow{PS}|}$

$= \dfrac{21}{\sqrt{50}\sqrt{18}}$

$QRS = \cos^{-1}\left(\dfrac{21}{\sqrt{50}\sqrt{18}}\right)$

$= 45.57$

$\boxed{QRS = 45.6^\circ}$

$\overrightarrow{PQ} \cdot \overrightarrow{PS} = (-3)(1) + (-4)(-1) + 5(4)$

$= (-3) + 4 + 20$

$= 21$

$|\overrightarrow{PQ}| = \sqrt{(-3)^2 + (-4)^2 + (5)^2}$

$= \sqrt{50}$

$|\overrightarrow{QR}| = \sqrt{(1)^2 + (-1)^2 + (4)^2}$

$= \sqrt{18}$

6. Solve $5\sin x - 4 = 2\cos 2x$ for $0 \le x < 2\pi$. **5**

$5\sin x - 4 = 2\cos 2x$

$5\sin x - 4 = 2(1 - 2\sin^2 x)$

$5\sin x - 4 = 2 - 4\sin^2 x$

$4\sin^2 x + 5\sin x - 6 = 0$

$(4\sin x - 3)(\sin x + 2) = 0$

$\sin x = \frac{3}{4}$ $\quad$ $\sin x = -2$

$\sin^{-1}\left(\frac{3}{4}\right) = 48.6°$

✓S	A✓
T	C

$\sin x = \frac{3}{4}$

$x = 48.6°, 131.4°$

$\boxed{x = 0.84,\ 2.29}$

$\sin x = -2$

$\boxed{\text{NO SOLS}}$

7. (a) Find the x-coordinate of the stationary point on the curve with equation $y = 6x - 2\sqrt{x^3}$. **4**

(b) Hence, determine the greatest and least values of y in the interval $1 \le x \le 9$. **3**

(a) $y = 6x - 2\sqrt{x^3}$

$= 6x - 2x^{3/2}$

$\frac{dy}{dx} = 6 - 3x^{1/2}$

$= 6 - 3\sqrt{x}$

FOR SP $\frac{dy}{dx} = 0$

$6 - 3\sqrt{x} = 0$

$3(2 - \sqrt{x}) = 0$

$2 - \sqrt{x} = 0$

$\sqrt{x} = 2$

$x = 4$

NATURE TABLE

x	$\rightarrow$	4	$\rightarrow$
$\frac{dy}{dx}$	+	0	−
SHAPE	/	—	\

$\boxed{x = 4 \text{ IS A MAX TP}}$

(b) WHEN $x = 1$

$y = 6(1) - 2\sqrt{(1)^3}$

$= 4$

$\boxed{\begin{array}{l}\text{WHEN } x = 4 \\ y = 6(4) - 2\sqrt{(4)^3} \\ \quad = 8 \\ \text{MAX VALUE}\end{array}}$

$\boxed{\begin{array}{l}\text{WHEN } x = 9 \\ y = 6(9) - 2\sqrt{(9)^3} \\ \quad = 0 \\ \text{MIN VALUE}\end{array}}$

8. Sequences may be generated by recurrence relations of the form $u_{n+1} = k u_n - 20$, $u_0 = 5$ where $k \in \mathbb{R}$.

(a) Show that $u_2 = 5k^2 - 20k - 20$. **2**

(b) Determine the range of values of k for which $u_2 < u_0$. **4**

(a) $u_1 = 5k - 20$

$u_2 = k(5k - 20) - 20$

$u_2 = 5k^2 - 20k - 20$

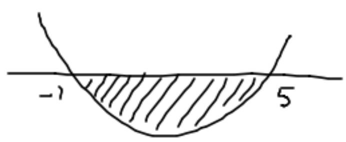
(b) $u_2 < u_0$

$5k^2 - 20k - 20 < 5$

$5k^2 - 20k - 25 < 0$

$5(k^2 - 4k - 5) < 0$

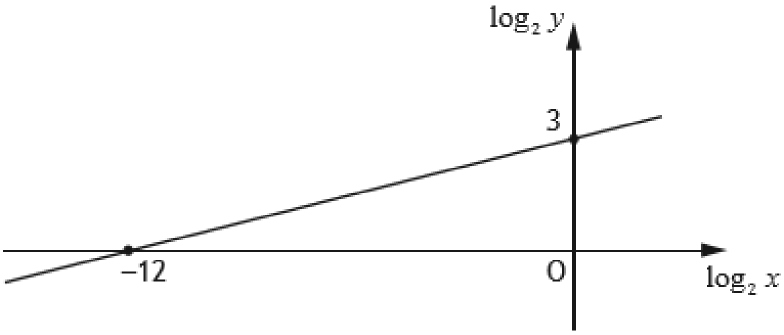
$5(k - 5)(k + 1) < 0$



$\boxed{-1 < k < 5}$

9. Two variables, x and y, are connected by the equation $y = kx^n$.

The graph of $\log_2 y$ against $\log_2 x$ is a straight line as shown.



Find the values of k and n. **5**

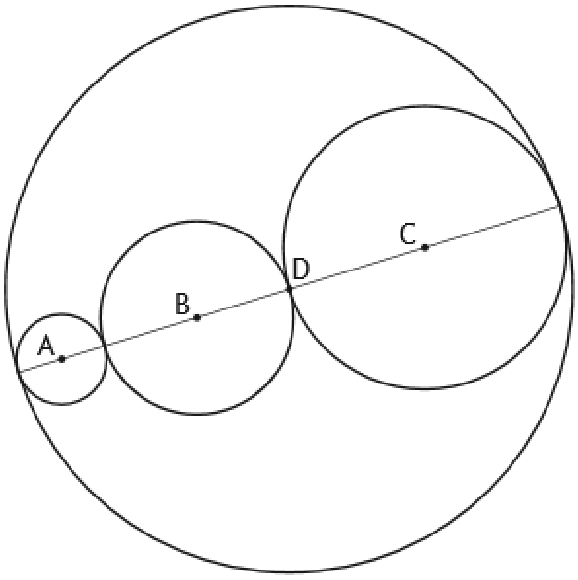
$$y = kx^n$$
$$\log_2 y = \log_2 kx^n$$
$$= \log_2 k + \log_2 x^n$$
$$= \log_2 k + n\log_2 x$$
$$= n\log_2 x + \log_2 k$$

$$n = m$$
$$n = \frac{3-0}{0-(-12)}$$
$$= \frac{3}{-12}$$
$$\boxed{n = -\frac{1}{4}}$$

$$\log_2 k = c$$
$$\log_2 k = 3$$
$$\boxed{k = 8}$$

10. (a) Show that the points A(−7, −2), B(2, 1) and C(17, 6) are collinear. **3**

Three circles with centres A, B and C are drawn inside a circle with centre D as shown.



The circles with centres A, B and C have radii r_A, r_B and r_C respectively.

- $r_A = \sqrt{10}$
- $r_B = 2r_A$
- $r_C = r_A + r_B$

(b) Determine the equation of the circle with centre D. **4**

(a) $m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-(-2)}{2-(-7)} = \frac{3}{9} = \frac{1}{3}$

$m_{BC} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-1}{17-2} = \frac{5}{15} = \frac{1}{3}$

$m_{AB} = m_{BC}$ SO LINES ARE PARALLEL SHARE COMMON POINT (B) SO A, B & C ARE COLLINEAR

(b) DIAMETER OF CIRCLE D

$= 2r_A + 2r_B + 2r_C$

$= 2r_A + 2(2r_A) + 2(r_A + r_B)$

$= 2r_A + 4r_A + 2r_2 + 2r_B$

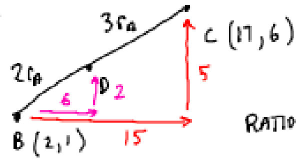
$= 2r_A + 4r_A + 2r_A + 4r_A$

$= 12r_A$

$= 12\sqrt{10}$

$r_D = 6\sqrt{10}$

CENTRE



RATIO 2 : 3

D (8,3)

CIRCLE D

$(x-8)^2 + (y-3) = (6\sqrt{10})^2$

$\boxed{(x-8)^2 + (y-3)^2 = 360}$

11. (a) Show that $\dfrac{\sin 2x}{2\cos x} - \sin x \cos^2 x = \sin^3 x$, where $0 < x < \dfrac{\pi}{2}$. **3**

(b) Hence, differentiate $\dfrac{\sin 2x}{2\cos x} - \sin x \cos^2 x$, where $0 < x < \dfrac{\pi}{2}$. **3**

(a)

$$\frac{\sin 2x}{2\cos x} - \sin x \cos^2 x$$

$$= \frac{2\sin x \cos x}{2\cos x} - \sin x \cos^2 x$$

$$= \sin x - \sin x \cos^2 x$$

$$= \sin x \left(1 - \cos^2 x\right)$$

$$= \sin x \ \sin^2 x$$

$$= \sin^3 x$$ AS REQUIRED

(b)

$$y = \frac{\sin 2x}{2\cos x} - \sin x \cos^2 x$$

$$= \sin^3 x$$

$$= (\sin x)^3$$

$$\frac{dy}{dx} = 3(\sin x)^2 \times \cos x$$

$$\boxed{\frac{dy}{dx} = 3\sin^2 x \cos x}$$