HIGHER MATHS

2016 Exam

Question Paper with Solutions

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2016 Non Calculator Paper

1. Find the equation of the line passing through the point $(-2, 3)$ which is parallel to the line with equation $y + 4x = 7$. **2**

$y = -4x + 7$

$\boxed{m = -4}$

$y - b = m(x - a)$

$y - 3 = -4(x + 2)$

$y - 3 = -4x - 8$

$\boxed{y = -4x - 5}$

2. Given that $y = 12x^3 + 8\sqrt{x}$, where $x > 0$, find $\dfrac{dy}{dx}$. **3**

$y = 12x^3 + 8\sqrt{x}$

$= 12x^3 + 8x^{1/2}$

$\dfrac{dy}{dx} = 36x^2 + 4x^{-1/2}$

$= 36x^2 + \dfrac{4}{\sqrt{x}}$

3. A sequence is defined by the recurrence relation $u_{n+1} = \frac{1}{3}u_n + 10$ with $u_3 = 6$.

(a) Find the value of u_4. **1**

(b) Explain why this sequence approaches a limit as $n \to \infty$. **1**

(c) Calculate this limit. **2**

(a) $u_{n+1} = \frac{1}{3}u_n + 10$

$u_4 = \frac{1}{3}u_3 + 10$

$= \frac{1}{3}(6) + 10$

$\boxed{u_4 = 12}$

(b) $\boxed{\text{LIMIT EXISTS AS } -1 < \tfrac{1}{3} < 1}$

(c) $L = \frac{1}{3}L + 10$

$L - \frac{1}{3}L = 10$

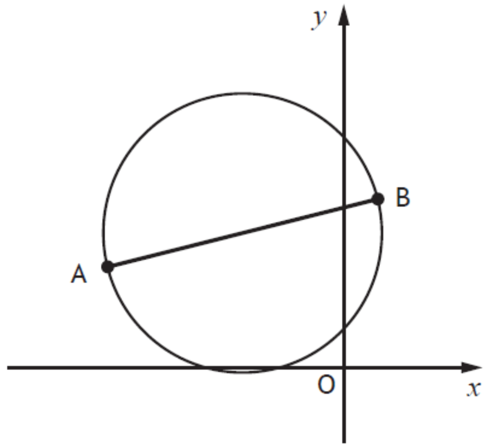
$\frac{2}{3}L = 10$

$2L = 30$

$\boxed{L = 15}$

4. A and B are the points $(-7, 3)$ and $(1, 5)$.

AB is a diameter of a circle.



Find the equation of this circle. **3**

CENTRE = MID POINT$_{AB}$ = $\left(\frac{-7+1}{2}, \frac{3+5}{2}\right)$

$= (-3, 4)$

DIAMETER $= \sqrt{(1-(-7))^2 + (5-3)^2}$

$= \sqrt{64 + 4}$

$= \sqrt{68}$

$= 2\sqrt{17}$ $\quad r = \sqrt{17}$

$\boxed{(x+3)^2 + (y-4)^2 = 17}$

3

5. Find $\int 8\cos(4x+1)\,dx$. **2**

$\int 8\cos(4x+1)\,dx$

$= \frac{1}{4} \times 8\sin(4x+1) + c$

$= 2\sin(4x+1) + c$

6. Functions f and g are defined on $\mathbb{R}$, the set of real numbers.

The inverse functions f^{-1} and g^{-1} both exist.

(a) Given $f(x) = 3x + 5$, find $f^{-1}(x)$. **3**

(b) If $g(2) = 7$, write down the value of $g^{-1}(7)$. **1**

(a)
$$y = 3x + 5$$
$$3x + 5 = y$$
$$3x = y - 5$$
$$x = \frac{y-5}{3}$$
$$\boxed{f^{-1}(x) = \frac{x-5}{3}}$$

(b)
$$g(2) = 7$$
$$\boxed{g^{-1}(7) = 2}$$

7. Three vectors can be expressed as follows:

$\overrightarrow{FG} = -2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$

$\overrightarrow{GH} = 3\mathbf{i} + 9\mathbf{j} - 7\mathbf{k}$

$\overrightarrow{EH} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

(a) Find $\overrightarrow{FH}$. **2**

(b) Hence, or otherwise, find $\overrightarrow{FE}$. **2**

(a)
$$\overrightarrow{FH} = \overrightarrow{FG} + \overrightarrow{GH}$$
$$= \begin{pmatrix} -2 \\ -6 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 9 \\ -7 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$$

(b)
$$\overrightarrow{FE} = \overrightarrow{FH} + \overrightarrow{HE}$$
$$= \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} + \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} \leftarrow \overrightarrow{EH} \times (-1)$$
$$= \begin{pmatrix} -1 \\ 0 \\ -5 \end{pmatrix}$$

8. Show that the line with equation $y = 3x - 5$ is a tangent to the circle with equation $x^2 + y^2 + 2x - 4y - 5 = 0$ and find the coordinates of the point of contact. **5**

$x^2 + y^2 + 2x - 4y - 5 = 0$

$x^2 + (3x-5)^2 + 2x - 4(3x-5) - 5 = 0$

$x^2 + 9x^2 - 30x + 25 + 2x - 12x + 20 - 5 = 0$

$10x^2 - 40x + 40 = 0$

$10(x^2 - 4x + 4) = 0$

$10(x-2)(x-2) = 0$

$x = 2 \quad x = 2$

REPEATED ROOT SO LINE IS A TANGENT

$y = 3(2) - 5$
$= 1$

POC $(2, 1)$

9. (a) Find the x-coordinates of the stationary points on the graph with equation $y = f(x)$, where $f(x) = x^3 + 3x^2 - 24x$. **4**

(b) Hence determine the range of values of x for which the function f is strictly increasing. **2**

(a) $f(x) = x^3 + 3x^2 - 24x$

$f'(x) = 3x^2 + 6x - 24$

FOR SP $f'(x) = 0$

$3x^2 + 6x - 24 = 0$

$3(x^2 + 2x - 8) = 0$

$3(x+4)(x-2) = 0$

$x = -4 \quad x = 2$

NATURE TABLE

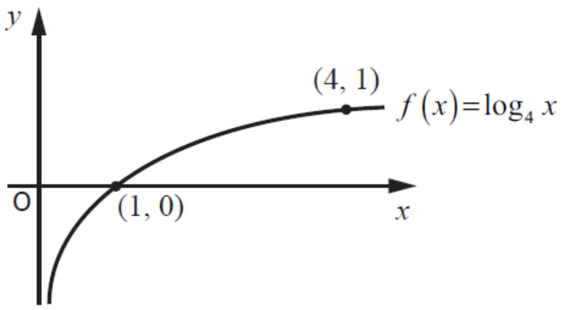
x	$\rightarrow$	-4	$\rightarrow$	2	$\rightarrow$
$f'(x)$	$+$	0	$-$	0	$+$
SHAPE	/	—	\	—	/

SP @ $x = -4$ AND $x = 2$

(b) $f(x)$ INCREASING WHEN $f'(x)$ IS $+$

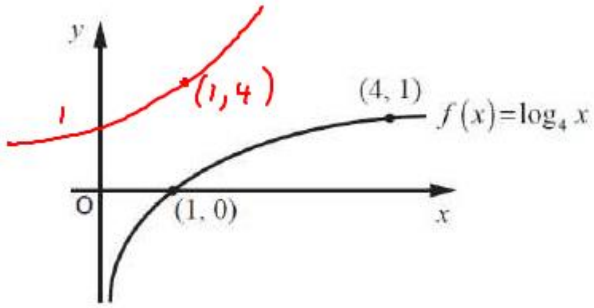
$x < -4$ AND $x > 2$

10. The diagram below shows the graph of the function $f(x) = \log_4 x$, where $x > 0$.



The inverse function, f^{-1}, exists.

On the diagram in your answer booklet, sketch the graph of the inverse function. **2**



11. (a) A and C are the points $(1, 3, -2)$ and $(4, -3, 4)$ respectively.

Point B divides AC in the ratio $1 : 2$.

Find the coordinates of B. **2**

(b) $k\overrightarrow{AC}$ is a vector of magnitude 1, where $k > 0$.

Determine the value of k. **3**

(a) $\overrightarrow{AC} = \underline{c} - \underline{a}$

$= \begin{pmatrix} 4 \\ -3 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ -6 \\ 6 \end{pmatrix}$

$\overrightarrow{AB} = \frac{1}{3}\overrightarrow{AC}$

$= \frac{1}{3}\begin{pmatrix} 3 \\ -6 \\ 6 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$B = A + \overrightarrow{AB}$

$= (1+1,\ 3+(-2),\ (-2)+2)$

$\boxed{B = (2, 1, 0)}$

(b) $|\overrightarrow{AC}| = \sqrt{(3)^2 + (-6)^2 + (6)^2}$

$= 9$

$|k\overrightarrow{AC}| = 1$

$\boxed{k = \frac{1}{9}}$

12. The functions f and g are defined on $\mathbb{R}$, the set of real numbers by

$f(x) = 2x^2 - 4x + 5$ and $g(x) = 3 - x$.

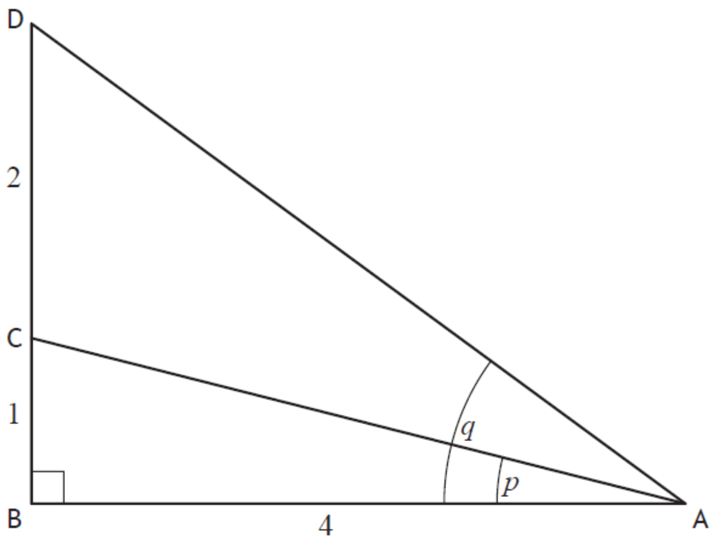
(a) Given $h(x) = f(g(x))$, show that $h(x) = 2x^2 - 8x + 11$. **2**

(b) Express $h(x)$ in the form $p(x+q)^2 + r$. **3**

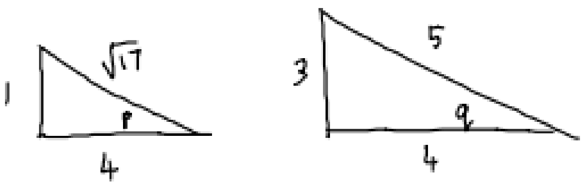
(a)
$$\begin{aligned} f(g(x)) &= f(3-x) \\ &= 2(3-x)^2 - 4(3-x) + 5 \\ &= 2(9 - 6x + x^2) - 12 + 4x + 5 \\ &= 18 - 12x + 2x^2 - 12 + 4x + 5 \\ &= 2x^2 - 8x + 11 \end{aligned}$$

(b)
$$\begin{aligned} &2x^2 - 8x + 11 \\ &= 2[x^2 - 4x] + 11 \\ &= 2[(x-2)^2 - 4] + 11 \\ &= 2(x-2)^2 - 8 + 11 \end{aligned}$$
$$\boxed{h(x) = 2(x-2)^2 + 3}$$

13. Triangle ABD is right-angled at B with angles BAC $= p$ and BAD $= q$ and lengths as shown in the diagram below.



Show that the exact value of $\cos(q-p)$ is $\dfrac{19\sqrt{17}}{85}$. **5**



$\sin p = \frac{O}{H} = \frac{1}{\sqrt{17}}$

$\cos p = \frac{A}{H} = \frac{4}{\sqrt{17}}$

$\sin q = \frac{O}{H} = \frac{3}{5}$

$\cos q = \frac{A}{H} = \frac{4}{5}$

$$\begin{aligned} &\cos(q-p) \\ &= \cos q \cos p + \sin q \sin p \\ &= \frac{4}{5} \times \frac{4}{\sqrt{17}} + \frac{3}{5} \times \frac{1}{\sqrt{17}} \\ &= \frac{16}{5\sqrt{17}} + \frac{3}{5\sqrt{17}} \\ &= \frac{19}{5\sqrt{17}} \quad \times \frac{\sqrt{17}}{\sqrt{17}} \\ &= \frac{19\sqrt{17}}{5 \times 17} \\ &= \frac{19\sqrt{17}}{85} \end{aligned}$$

AS REQUIRED

14. (a) Evaluate $\log_5 25$. **1**

(b) Hence solve $\log_4 x + \log_4 (x-6) = \log_5 25$, where $x > 6$. **5**

(a) $\log_5 25 = 2$

(b) $\log_4 x + \log_4 (x-6) = \log_5 25$

$$\log_4 x(x-6) = 2$$

$$x(x-6) = 4^2$$

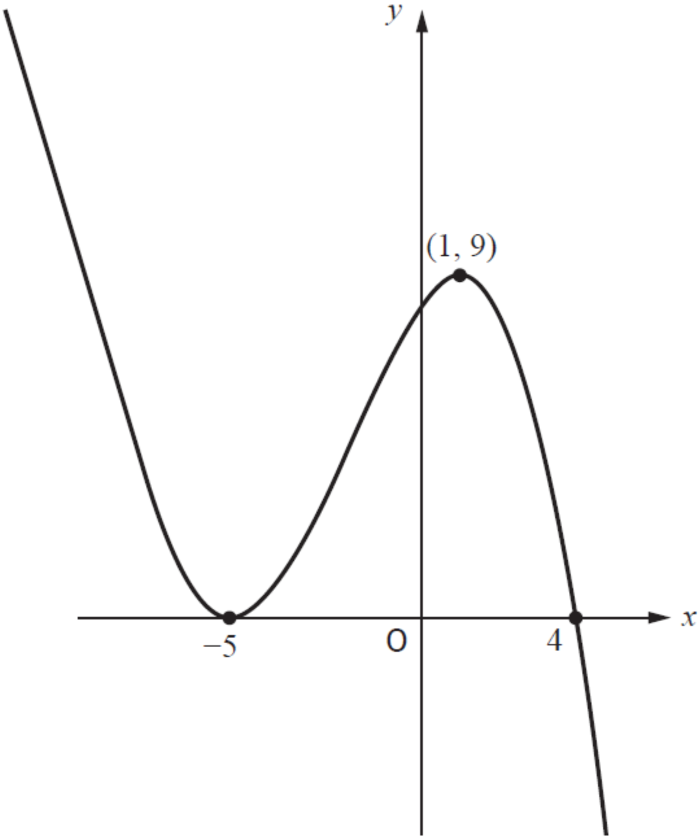
$$x^2 - 6x = 16$$

$$x^2 - 6x - 16 = 0$$

$$(x-8)(x+2) = 0$$

$$\boxed{x = 8} \quad \cancel{x = -2}$$

15. The diagram below shows the graph with equation $y = f(x)$, where $f(x) = k(x-a)(x-b)^2$.



(a) Find the values of a, b and k. **3**

(b) For the function $g(x) = f(x) - d$, where d is positive, determine the range of values of d for which $g(x)$ has exactly one real root. **1**

(a) $f(x) = k(x-a)(x-b)^2$

$= k(x-4)(x+5)^2$

$9 = k(1-4)(1+5)^2$

$= k(-3)(36)$

$= -108k$

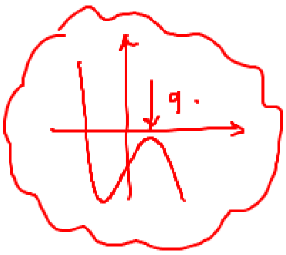
$k = -\frac{1}{12}$

$\boxed{f(x) = -\frac{1}{12}(x-4)(x+5)^2}$

(b) $f(x) - d$

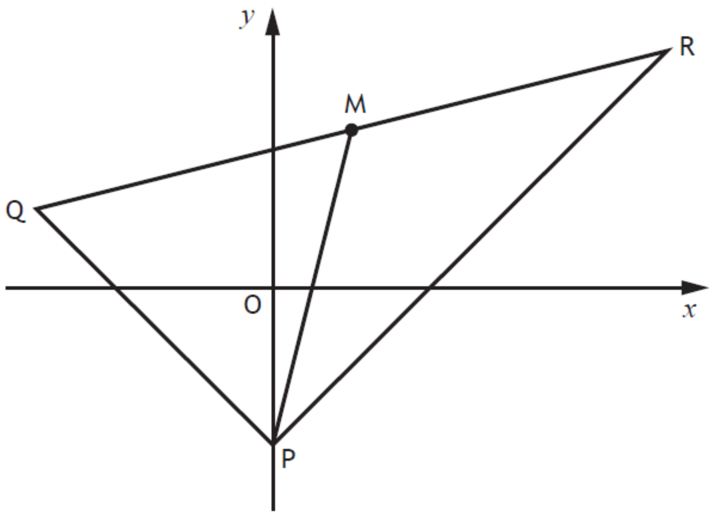
moves $f(x)$ ↓

$\boxed{d > 9}$



2016 Calculator Paper

1. PQR is a triangle with vertices P$(0,-4)$, Q$(-6,2)$ and R$(10,6)$.



(a) (i) State the coordinates of M, the midpoint of QR. **1**

(ii) Hence find the equation of PM, the median through P. **2**

(b) Find the equation of the line, *L*, passing through M and perpendicular to PR. **3**

(c) Show that line *L* passes through the midpoint of PR. **3**

(a)(i) $M = \left(\frac{-6+10}{2}, \frac{2+6}{2}\right)$

$= (2,4)$

(ii) $m_{PM} = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{4-(-4)}{2-0}$

$= \frac{8}{2}$

$= 4$

$y - b = m(x-a)$

$y - 4 = 4(x-2)$

$y - 4 = 4x - 8$

$\boxed{y = 4x - 4}$

(b) $m_{PR} = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{-4-6}{0-10}$

$= \frac{-10}{-10}$

$= 1$

$m_1 \times m_2 = -1$

$m_\perp = -1$

$y - b = m(x-a)$

$y - 4 = -1(x-2)$

$y - 4 = -x + 2$

$\boxed{y = -x + 6}$

(c) Mid PR $= \left(\frac{0+10}{2}, \frac{-4+6}{2}\right)$

$= (5,1)$

$y = -(5) + 6$

$= 1$

$\boxed{\text{So } (5,1) \text{ LIES ON LINE } y = -x+6}$

2. Find the range of values for p such that $x^2 - 2x + 3 - p = 0$ has no real roots. **3**

FOR NO REAL ROOTS $b^2 - 4ac < 0$

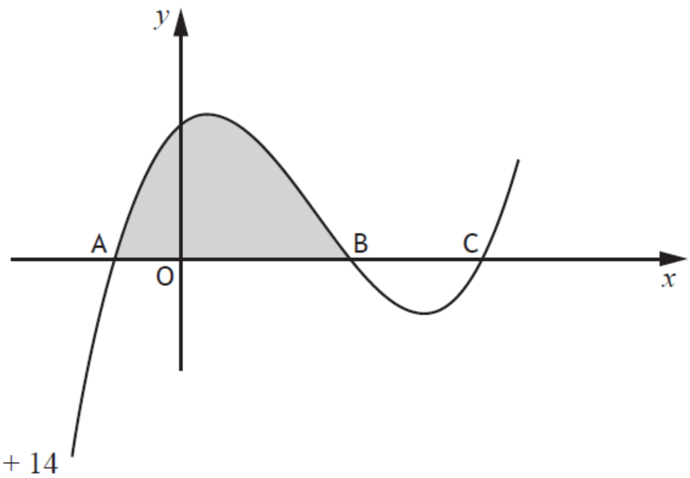
$$(-2)^2 - (4 \times 1 \times (3-p)) < 0$$
$$4 - (4(3-p)) < 0$$
$$4 - 12 + 4p < 0$$
$$-8 + 4p < 0$$
$$4p < 8$$
$$\boxed{p < 2}$$

3. (a) (i) Show that $(x+1)$ is a factor of $2x^3 - 9x^2 + 3x + 14$. **2**

(ii) Hence solve the equation $2x^3 - 9x^2 + 3x + 14 = 0$. **3**

(b) The diagram below shows the graph with equation $y = 2x^3 - 9x^2 + 3x + 14$.

The curve cuts the x-axis at A, B and C.



(i) Write down the coordinates of the points A and B. **1**

(ii) Hence calculate the shaded area in the diagram. **4**

(a)(i) -1

-1	2	-9	3	14
		-2	11	-14
	2	-11	14	$\boxed{0}$

REM = 0 SO $x+1$ IS A FACTOR

(ii) $2x^3 - 9x^2 + 3x + 14 = 0$

$(x+1)(2x^2 - 11x + 14) = 0$

$(x+1)(2x-7)(x-2) = 0$

$\boxed{x = -1 \quad x = \frac{7}{2} \quad x = 2}$

(b)(i) $\boxed{A\ (-1,0) \quad B = (2,0)}$

(b)(ii) $\int_{-1}^{2} 2x^3 - 9x^2 + 3x + 14 \, dx$

$= \left[\frac{2x^4}{4} - \frac{9x^3}{3} + \frac{3x^2}{2} + 14x \right]_{-1}^{2}$

$= \left[\frac{x^4}{2} - 3x^3 + \frac{3x^2}{2} + 14x \right]_{-1}^{2}$

$= \left(\frac{(2)^4}{2} - 3(2)^3 + \frac{3(2)^2}{2} + 14(2) \right) - \left(\frac{(-1)^4}{2} - 3(-1)^3 + \frac{3(-1)^2}{2} + 14(-1) \right)$

$= (8 - 24 + 6 + 28) - \left(\frac{1}{2} + 3 + \frac{3}{2} - 14 \right)$

$= 18 - (-9)$

$= 27 \text{ units}^2$

4. Circles C_1 and C_2 have equations $(x+5)^2+(y-6)^2=9$ and $x^2+y^2-6x-16=0$ respectively.

(a) Write down the centres and radii of C_1 and C_2. **4**

(b) Show that C_1 and C_2 do not intersect. **3**

(a) C_1

CENTRE $= (-5, 6)$

RADIUS $= 3$

C_2

CENTRE $(3, 0)$

RADIUS $= \sqrt{(3)^2+(0)^2-(-16)}$

$= 5$

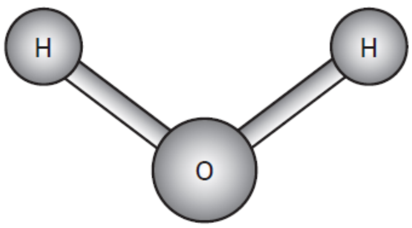
(b) DISTANCE BETWEEN CENTRES $= \sqrt{(-5-3)^2+(6-0)^2}$

$= \sqrt{100}$

$= 10$

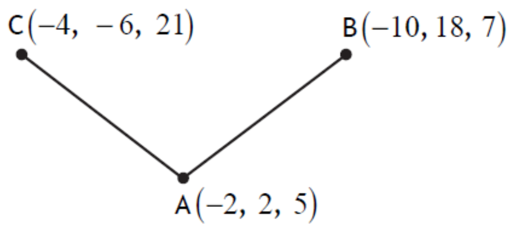
$\boxed{\text{BECAUSE } D > r_1 + r_2 \ (10 > 3+5) \text{ CIRCLES } \underline{\text{DON'T}} \text{ INTERSECT}}$

5. The picture shows a model of a water molecule.



Relative to suitable coordinate axes, the oxygen atom is positioned at point A$(-2, 2, 5)$.

The two hydrogen atoms are positioned at points B$(-10, 18, 7)$ and C$(-4, -6, 21)$ as shown in the diagram below.



(a) Express $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in component form. **2**

(b) Hence, or otherwise, find the size of angle BAC. **4**

(a) $\overrightarrow{AB} = \underline{b} - \underline{a}$

$= \begin{pmatrix}-10\\18\\7\end{pmatrix} - \begin{pmatrix}-2\\2\\5\end{pmatrix}$

$= \begin{pmatrix}-8\\16\\2\end{pmatrix}$

$\overrightarrow{AC} = \underline{c} - \underline{a}$

$= \begin{pmatrix}-4\\-6\\21\end{pmatrix} - \begin{pmatrix}-2\\2\\5\end{pmatrix}$

$= \begin{pmatrix}-2\\-8\\16\end{pmatrix}$

(b) $\cos BAC = \dfrac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}$

$\overrightarrow{AB}\cdot\overrightarrow{AC} = (-8)(-2) + 16(-8) + 2(16)$

$= -80$

$|\overrightarrow{AB}| = \sqrt{(-8)^2+(16)^2+(2)^2}$

$= \sqrt{324} = 18$

$\overrightarrow{AC} = \sqrt{(-2)^2+(-8)^2+(16)^2}$

$= \sqrt{324} = 18$

$\cos BAC = \dfrac{-80}{18(18)}$

$BAC = \cos^{-1}\left(\dfrac{-80}{324}\right)$

$\boxed{BAC = 104.3^\circ}$

6. Scientists are studying the growth of a strain of bacteria. The number of bacteria present is given by the formula

$$B(t) = 200e^{0\cdot107t},$$

where t represents the number of hours since the study began.

(a) State the number of bacteria present at the start of the study. **1**

(b) Calculate the time taken for the number of bacteria to double. **4**

(a) $B(0) = 200e^{0.107(0)}$

$= 200e^{0}$

$\boxed{B(0) = 200}$

(b) $400 = 200e^{0.107t}$

$2 = e^{0.107t}$

$\ln 2 = \ln e^{0.107t}$

$\ln 2 = 0.107t \ln e$

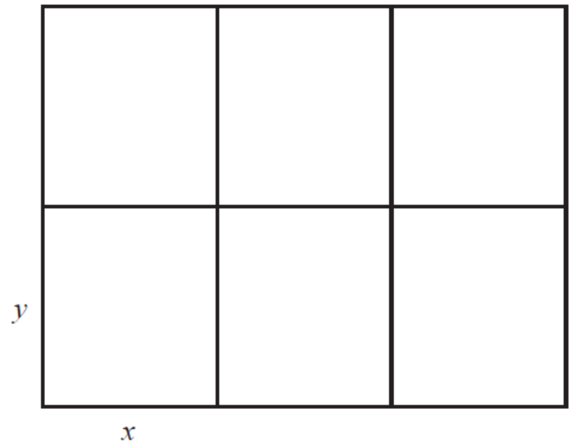
$0.107t = \ln 2$

$t = \dfrac{\ln 2}{0.107}$

$\boxed{t = 6.478 \text{ hours}}$

7. A council is setting aside an area of land to create six fenced plots where local residents can grow their own food.

Each plot will be a rectangle measuring x metres by y metres as shown in the diagram.



(a) The area of land being set aside is $108\ \text{m}^2$.

Show that the total length of fencing, L metres, is given by

$$L(x) = 9x + \frac{144}{x}.$$ **3**

(b) Find the value of x that minimises the length of fencing required. **6**

(a) AREA

$3x \times 2y = 108$

$6xy = 108$

$y = \frac{18}{x}$

LENGTH

$9x + 8y$

$= 9x + 8\left(\frac{18}{x}\right)$

$= 9x + \frac{144}{x}$

(b) $L(x) = 9x + \frac{144}{x}$

$= 9x + 144x^{-1}$

$L'(x) = 9 - 144x^{-2}$

$= 9 - \frac{144}{x^2}$

FOR SP $L'(x) = 0$

$9 - \frac{144}{x^2} = 0$

$9 = \frac{144}{x^2}$

$9x^2 = 144$

$x^2 = 16$

$x = \pm 4$

NATURE TABLE

x	$\to$	4	$\to$
$L'(x)$	$-$	0	$+$
SHAPE	\	—	/

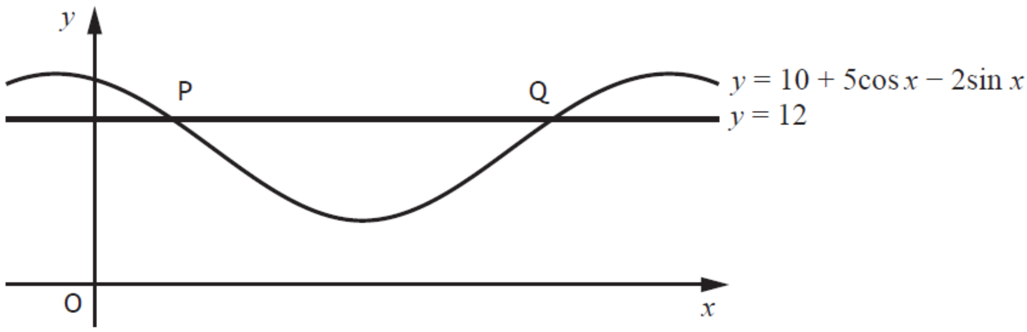
$\boxed{\text{MIN TP WHEN } x = 4}$

8. (a) Express $5\cos x - 2\sin x$ in the form $k\cos(x+a)$,

where $k > 0$ and $0 < a < 2\pi$. **4**

(b) The diagram shows a sketch of part of the graph of $y = 10 + 5\cos x - 2\sin x$ and the line with equation $y = 12$.

The line cuts the curve at the points P and Q.



Find the x-coordinates of P and Q. **4**

(a) $5\cos x - 2\sin x = k\cos(x+a)$

$= k\cos x\cos a - k\sin x\sin a$

$-k\sin a = -2$

$k\sin a = 2$

$k\cos a = 5$

$\tan a = \frac{2}{5}$

S	A
T	C

$\tan^{-1}\left(\frac{2}{5}\right)$

$= 21.8°$

$= 0.38$ rads

$a = 0.38$

$k = \sqrt{(5)^2 + (2)^2}$

$= \sqrt{29}$

4

$5\cos x - 2\sin x = \sqrt{29}\cos(x + 0.38)$

(b) $10 + \sqrt{29}\cos(x + 0.38) = 12$

$\sqrt{29}\cos(x + 0.38) = 2$

$\cos(x + 0.38) = \frac{2}{\sqrt{29}}$

$\cos^{-1}\left(\frac{2}{\sqrt{29}}\right)$

$= 68.1°$

$= 1.19$ rads

S	A
T	C

$x + 0.38 = 1.19, 5.09$

$x = 0.81, 4.71$

$\boxed{P\ (0.81, 12) \quad Q\ (4.71, 12)}$

9. For a function f, defined on a suitable domain, it is known that:

- $f'(x) = \dfrac{2x+1}{\sqrt{x}}$

- $f(9) = 40$

Express $f(x)$ in terms of x. **4**

$$f'(x) = \frac{2x+1}{\sqrt{x}}$$
$$= \frac{2x}{x^{1/2}} + \frac{1}{x^{1/2}}$$
$$= 2x^{1/2} + x^{-1/2}$$

$$f'(x) = 2x^{1/2} + x^{-1/2}$$
$$f(x) = \frac{2x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C$$
$$= \frac{4\sqrt{x^3}}{3} + 2\sqrt{x} + C$$

$$40 = \frac{4\sqrt{(9)}^3}{3} + 2\sqrt{(9)} + C$$
$$40 = 36 + 6 + C$$
$$40 = 42 + C$$
$$C = -2$$
$$f(x) = \frac{4\sqrt{x}^3}{3} + 2\sqrt{x} - 2$$

10. (a) Given that $y = \left(x^2+7\right)^{\frac{1}{2}}$, find $\dfrac{dy}{dx}$. **2**

(b) Hence find $\displaystyle\int \frac{4x}{\sqrt{x^2+7}}\,dx$. **1**

(a) $y = \left(x^2+7\right)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}\left(x^2+7\right)^{-1/2} \times 2x$$
$$= \frac{x}{\sqrt{x^2+7}}$$

(b) $\displaystyle\int \frac{4}{\sqrt{x^2+7}}\,dx$

$$= \int 4\left(x^2+7\right)^{-1/2}\,dx$$
$$= 4\left(x^2+7\right)^{1/2} + C$$

11. (a) Show that $\sin 2x \tan x = 1 - \cos 2x$, where $\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$. **4**

(b) Given that $f(x) = \sin 2x \tan x$, find $f'(x)$. **2**

(a)
$$\sin 2x \tan x = 1 - \cos 2x$$
$$2\sin x\cos x \tan x = 1 - (1 - 2\sin^2 x)$$
$$2\sin x\cos x \frac{\sin x}{\cos x} = 2\sin^2 x$$
$$2\sin^2 x = 2\sin^2 x$$

AS REQUIRED.

(b)
$$f(x) = \sin 2x \tan x$$
$$= 2\sin^2 x$$
$$= 2(\sin x)^2$$
$$f'(x) = 4(\sin x) \times -\cos x$$
$$\boxed{f'(x) = -4\sin x\cos x}$$