HIGHER MATHS

Wave Function

Notes with Examples

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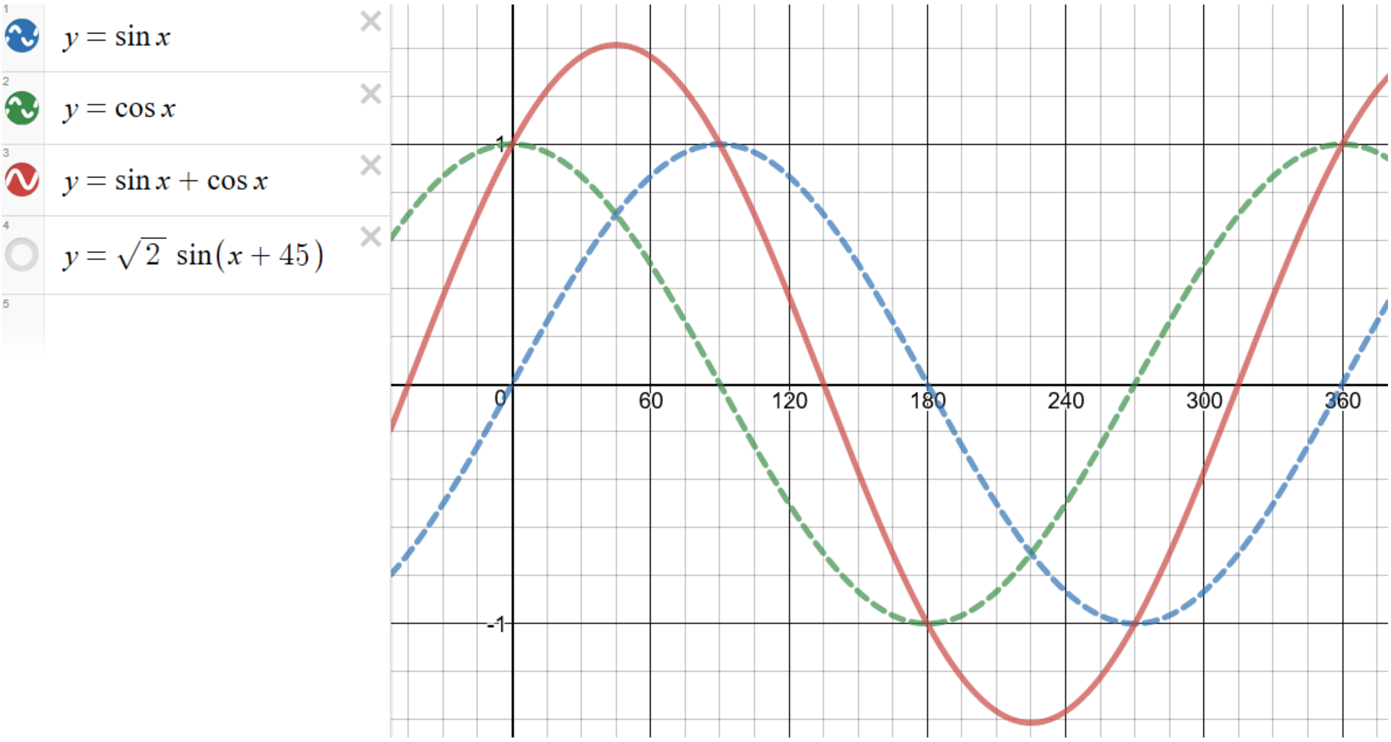
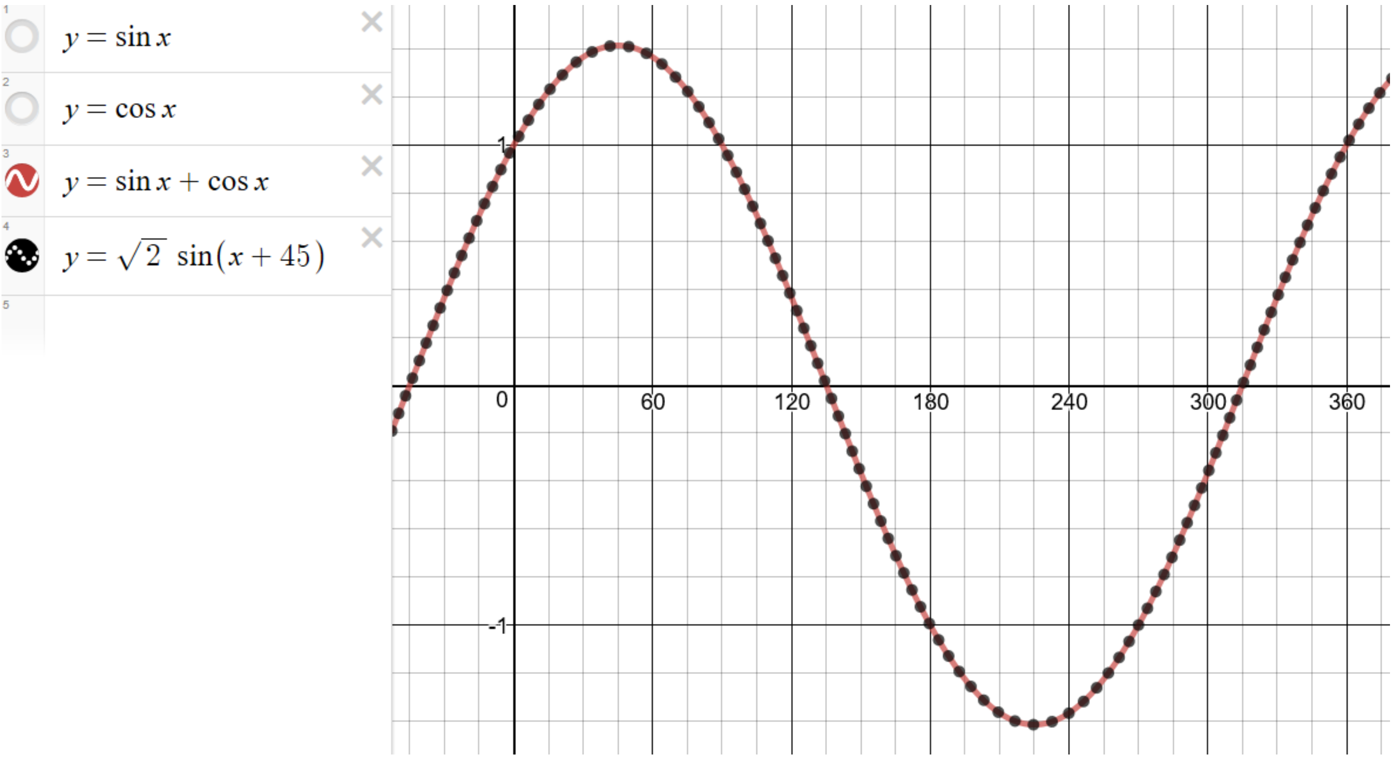
The Wave Function

The wave function is the addition (or difference) of two waves into a single wave.



So $a\sin x + \mathrm{b}\cos x$ can be written in the form of $k\sin(x \pm \alpha)$ or$k\cos(x \pm \alpha)$.



To solve these types of questions we follow these steps.

* compare $a\sin x + \text{b}\cos x$ with the expanded from of $k\sin(x \pm \alpha)$ or $k\cos(x \pm \alpha)$
* list the values of $k\sin\propto$ and $k\cos\propto$
* calculate k
* calculate $\propto$
* state the answer in form $k\sin(x \pm \alpha)$ or $k\cos(x \pm \alpha)$

Worked Example

Express $2\sin x - 5\cos x$ in the form $k\sin(x-\alpha)$, where $k > 0$ and $0° \leq\propto\leq 360°$

$$2\sin x - 5\cos x = k\sin(x-\alpha)$$
$$= k\sin x\cos\alpha - k\cos x\sin\alpha$$

$-k\sin\alpha = -5$

$k\sin\alpha = 5$

$k\cos\alpha = 2$

$\tan\alpha = \frac{5}{2}$

$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$

$\alpha = \tan^{-1}\left(\frac{5}{2}\right)$

$= 68.2^{\circ}$

S	A
T	C

$k = \sqrt{2^2 + 5^2}$

$= \sqrt{29}$

$$2\sin x - 5\cos x = \sqrt{29}\sin(x - 68.2)^{\circ}$$

Examples

W-01 Express $\cos x - 3\sin x$ in the form $k\cos(x-\propto)$, $0° \leq\propto\leq 360°$.

$$\cos x - 3\sin x = k\cos(x-\alpha)$$
$$= k\cos x\cos\alpha + k\sin x\sin\alpha$$

$k\sin\alpha = -3$

$k\cos\alpha = 1$

$\tan\alpha = \frac{-3}{1} = -3$

$\tan^{-1}(3) = 71.6^{\circ}$

S	A
T	C

$\alpha = 360 - 71.6$

$= 288.4^{\circ}$

$k = \sqrt{(-3)^2 + (1)^2}$

$= \sqrt{10}$

$$\cos x - 3\sin x = \sqrt{10}\cos(x - 288.4)^{\circ}$$

W-02 Write $4\cos x - 3\sin x$ in the form $k\sin(x-\alpha)$, $0° \leq \alpha \leq 360°$.

$$4\cos x - 3\sin x = k\sin(x-\alpha)$$
$$= k\sin x\cos\alpha - k\cos x\sin\alpha$$

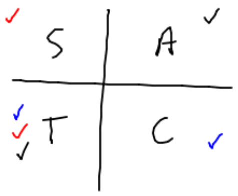
$-k\sin\alpha = 4$

$k\sin\alpha = -4$

$k\cos\alpha = -3$

$\tan\alpha = \frac{-4}{-3} = \frac{4}{3}$

$\tan^{-1}\left(\frac{4}{3}\right) = 53.1°$



$\alpha = 180 + 53.1$
$= 233.1°$

$k = \sqrt{(-4)^2 + (-3)^2}$
$= 5$

$4\cos x - 3\sin x = 5\sin(x - 233.1)°$

W-03 Express $\sin x - \cos x$ in the form $k\sin(x+\alpha)$, $0° \leq \alpha \leq 360°$.

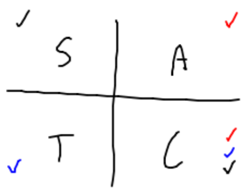
$$\sin x - \cos x = k\sin(x+\alpha)$$
$$= k\sin x\cos\alpha + k\cos x\sin\alpha$$

$k\sin\alpha = -1$

$k\cos\alpha = 1$

$\tan\alpha = \frac{-1}{1} = -1$

$\tan^{-1}(1) = 45°$



$\alpha = 360 - 45$
$= 315°$

$k = \sqrt{(1)^2 + (-1)^2}$
$= \sqrt{2}$

$\sin x - \cos x = \sqrt{2}\sin(x - 315)°$

The Wave Function with Radians and Compound Angles

We can use the same method as previously described to work in radians and with compound angles.

Examples

W-04 Write $5\cos 3x - 12\sin 3x$ in the form $k\sin(3x - \alpha)$, $0° \leq \alpha \leq 360°$.

$$5\cos 3x - 12\sin 3x = k\sin(3x-\alpha)$$
$$= k\sin 3x\cos\alpha - k\cos 3x\sin\alpha$$

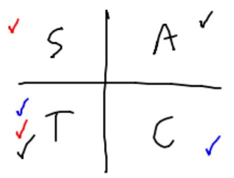
$-k\sin\alpha = 5$

$k\sin\alpha = -5$

$k\cos\alpha = -12$

$\tan\alpha = \frac{-5}{-12} = \frac{5}{12}$

$\tan^{-1}\left(\frac{5}{12}\right) = 22.6°$



$\alpha = 180 + 22.6$

$= 202.6°$

$k = \sqrt{(-5)^2 + (-12)^2}$

$= 13$

$5\cos 3x - 12\sin 3x = 13\sin(3x - 202.6)°$

W-05 Express $2\sqrt{3}\cos x + 2\sin x$ in the form $k\sin(x + \alpha)$, $0 \leq \alpha \leq 2\pi$.

$$2\sqrt{3}\cos x + 2\sin x = k\sin(x-\alpha)$$
$$= k\sin x\cos\alpha - k\cos x\sin\alpha$$

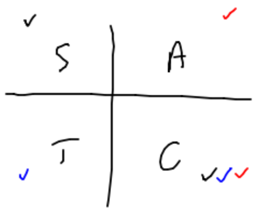
$-k\sin\alpha = 2\sqrt{3}$

$k\sin\alpha = -2\sqrt{3}$

$k\cos\alpha = 2$

$\tan\alpha = \frac{-2\sqrt{3}}{2} = -\sqrt{3}$

$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$



$k = \sqrt{(2\sqrt{3})^2 + 2^2}$

$= 4$

$\alpha = 2\pi - \frac{\pi}{3}$

$= \frac{6\pi}{3} - \frac{\pi}{3}$

$= \frac{5\pi}{3}$

$2\sqrt{3}\cos x + 2\sin x = 4\sin\left(x - \frac{5\pi}{3}\right)$

Maxima and Minima of $a \sin x \pm \text{b}\cos x$

To calculate the maximum and minimum value of $a \sin x + \text{b}\cos x$ we first have to write it in the form $k \sin(x \pm \alpha)$ or $k \cos(x \pm \alpha)$. Then we use the single function to determine the max and min values and when they occur (think back to trig graphs).

Examples

W-06 State the maximum and minimum values of $\text{y} = 4\cos x + 3\sin x$ for $0° \leq \alpha \leq 360°$, and the corresponding values of x.

$$\boxed{\begin{array}{l} \text{MAX} = 5 \ @ \ x + 53.1 = 90 \\ \qquad\qquad\qquad x = 36.9° \\ \text{MIN} = -5 \ @ \ x + 53.1 = 270 \\ \qquad\qquad\qquad = 216.9° \end{array}}$$

$$4\cos x + 3\sin x = k\sin(x + \alpha)$$
$$= k\sin x\cos\alpha + k\cos x\sin\alpha$$

$k\sin\alpha = 4$

$k\cos\alpha = 3$

$\tan\alpha = \frac{4}{3}$

$\tan^{-1}\left(\frac{4}{3}\right) = 53.1°$

S	A ✓✓
✓T	C ✓

$\alpha = 53.1°$

$k = \sqrt{4^2 + 3^2}$
$= 5$

$4\cos x + 3\sin x = 5\sin(x + 53.1)°$

W-07 State the maximum and minimum values of $\text{y} = 7\sin x + 4\cos x + 12$ for $0° \leq \alpha \leq 360°$, and the corresponding values of x.

$$7\sin x + 4\cos x = k\cos(x - \alpha)$$
$$= k\cos x\cos\alpha + k\sin x\sin\alpha$$

$k\sin\alpha = 7$

$k\cos\alpha = 4$

$\tan\alpha = \frac{7}{4}$

$\tan^{-1}\left(\frac{7}{4}\right) = 60.3$

✓ S	A ✓✓
✓T	C ✓

$\alpha = 60.3$

$k = \sqrt{7^2 + 4^2}$
$= \sqrt{65}$

$7\sin x + 4\cos x = \sqrt{65}\cos(x - 60.3)$

$y = 7\sin x + 4\cos x + 12$
$= \sqrt{65}\cos(x - 60.3) + 12$

$\text{MAX} = \sqrt{65} + 12 \ @ \ x - 60.3 = 0$
$x = 60.3°$

$\text{MIN} = -\sqrt{65} + 12 \ @ \ x - 60.3 = 180$
$x = 240.3°$

Solving Equations using the Wave Function

We can use the wave function to solve trigonometric equations.

Examples

W-08 Solve $\cos x - 3\sin x = 3$ for $0° \leq x \leq 360°$.

$$\cos x - 3\sin x = k\cos(x+\alpha)$$
$$= k\cos x\cos\alpha - k\sin x\sin\alpha$$

$-k\sin\alpha = -3$

$k\sin\alpha = 3$

$k\cos\alpha = 1$

$\tan\alpha = \frac{3}{1} = 3$

$\tan^{-1}(3) = 71.6°$

✓ S	A ✓✓✓
✓ T	C ✓

$\alpha = 71.6°$

$k = \sqrt{(-3)^2 + 1^2}$

$= \sqrt{10}$

$\cos x - 3\sin x = \sqrt{10}\cos(x + 71.6)°$

$\sqrt{10}\cos(x+71.6) = 3$

$\cos(x+71.6) = \frac{3}{\sqrt{10}}$

S	A ✓
T	C ✓

$x + 71.6 = 18.5,\ 341.5,\ 378.5$

$x = \cancel{-53.1},\ 269.9,\ 6.9$

W-09 The blades of a wind turbine are turning at a steady rate.

The height, h metres, of the tip of one of the blades above the ground at time, t seconds, is given by the formula

$$h = 36\sin(1\cdot5t) - 15\cos(1\cdot5t) + 65.$$

Express $36\sin(1\cdot5t) - 15\cos(1\cdot5t)$ in the form

$$k\sin(1\cdot5t - a), \text{ where } k > 0 \text{ and } 0 < a < \frac{\pi}{2},$$

and hence find the **two** values of t for which the tip of this blade is at a height of 100 metres above the ground during the first turn.

$$36\sin(1.5t) - 15\cos(1.5t) = k\sin(1.5t - \alpha)$$
$$= k\sin 1.5t\cos\alpha - k\cos 1.5t\sin\alpha$$

$-k\sin\alpha = -15$

$k\sin\alpha = 15$

$k\cos\alpha = 36$

$\tan\alpha = \left(\frac{15}{36}\right)$

$\tan^{-1}\left(\frac{15}{36}\right) = 22.6^\circ$

✓ S	A ✓ ✓
✓ T	C ✓

$k = \sqrt{36^2 + (-15)^2}$

$= 39$

$36\sin(1.5t) - 15\cos(1.5t) = 39\sin(1.5t - 22.6)^\circ$

$$h = 36\sin(1.5t) - 15\cos(1.5t) + 65$$
$$= 39\sin(1.5t - 22.6)^\circ + 65$$

FOR HEIGHT = 100

$$39\sin(1.5t - 22.6)^\circ + 65 = 100$$
$$39\sin(1.5t - 22.6)^\circ = 35$$
$$\sin(1.5t - 22.6)^\circ = \frac{35}{39}$$

$\sin^{-1}\left(\frac{35}{39}\right) = 63.8^\circ$

✓ S	A ✓
T	C

$$(1.5t - 22.6)^\circ = 63.8^\circ, 116.2^\circ$$
$$1.5t = 86.4, 138.8^\circ$$
$$t = 57.6, 92.5^\circ$$
$$t = 1.005, 1.615 \text{ radians}$$

Summary

Wave Function

allows use to add a cos and sin wave.
uses the compound angle formulae.

$$a\cos x + b\sin x = k\cos(x-\alpha)$$

Could be:
$k\cos(x \pm \alpha)$
$k\sin(x \pm \alpha)$

Follow this example:

$$4\sin x - 3\cos x = k\sin(x-\alpha)$$
$$= k\sin x\cos\alpha - k\cos x\sin\alpha$$

$-k\sin\alpha = -3$ so $k\sin\alpha = 3$

$k\cos\alpha = 4$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{3}{4}$$

$$\begin{array}{c|c} \checkmark\ S & A\ \checkmark\checkmark\checkmark \\ \hline \checkmark\ T & C\ \checkmark \end{array}$$

$\alpha = 36.9°$

$k = \sqrt{3^2 + 4^2}$ ← from $k\sin\alpha$ and $k\cos\alpha$

$= \sqrt{25}$

$= 5$

finish with this statement →

$$4\sin x - 3\cos x = 5\sin(x - 36.9)°$$

You could now use wave function to solve $4\sin x - 3\cos x = 1$ by rewriting to $5\sin(x-36.9)° = 1$