HIGHER MATHS

Vectors

Notes with Examples

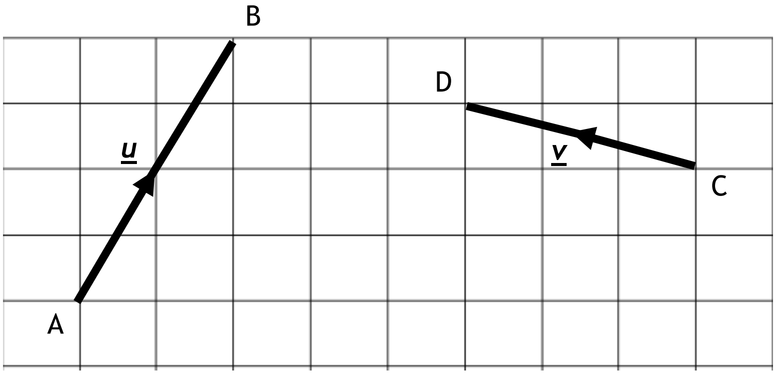
Mr Miscandlon
Gw13miscandlondavid@glow.sch.uk

Recap on National 5 Vectors

A vector has both magnitude (size) and direction.

Naming a Vector

A vector is named either using the letters at each end of the directed line segment or a single bold or underlined letter.



$\overrightarrow{AB}$ or $\underline{\mathbf{u}}$ in the diagram is a directed line segment.

$\begin{pmatrix}2\\4\end{pmatrix}$ is $\overrightarrow{AB}$ in **component form.**

Adding, Subtracting and Multiplying by a Scalar

$\underline{\mathbf{u}} + \underline{\mathbf{v}}$

$= \begin{pmatrix}2\\4\end{pmatrix} + \begin{pmatrix}-3\\1\end{pmatrix}$

$= \begin{pmatrix}-1\\5\end{pmatrix}$

$\underline{\mathbf{u}} - \underline{\mathbf{v}}$

$= \begin{pmatrix}2\\4\end{pmatrix} - \begin{pmatrix}-3\\1\end{pmatrix}$

$= \begin{pmatrix}5\\3\end{pmatrix}$

$3\underline{\mathbf{u}} - 4\underline{\mathbf{v}}$

$= 3\begin{pmatrix}2\\4\end{pmatrix} - 4\begin{pmatrix}-3\\1\end{pmatrix}$

$= \begin{pmatrix}6\\12\end{pmatrix} - \begin{pmatrix}-12\\4\end{pmatrix}$

$= \begin{pmatrix}18\\8\end{pmatrix}$

NEVER try to simplify a vector like a fraction

Magnitude of a Vector

The size of a vector can be calculated by squaring each component, adding together then square rooting.

$$\overrightarrow{AB} = \begin{pmatrix}2\\4\end{pmatrix}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{2^2 + 4^2}$$

$$= \sqrt{20}$$

$$= 2\sqrt{5}$$

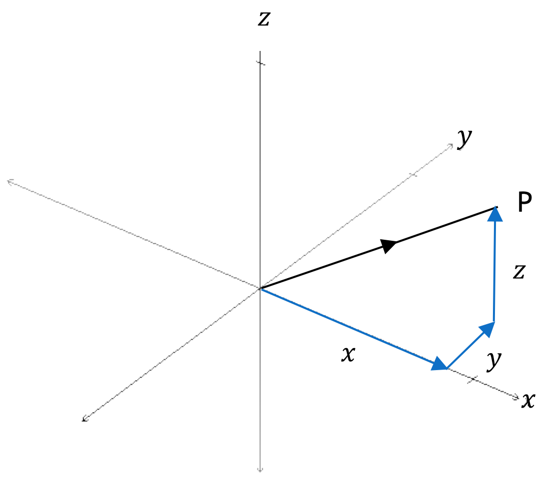
$$\overrightarrow{PQ} = \begin{pmatrix}3\\-1\\2\end{pmatrix}$$

$$\left|\overrightarrow{PQ}\right| = \sqrt{3^2 + (-1)^2 + 2^2}$$

$$= \sqrt{14}$$

Position Vector

$\overrightarrow{OP}$ is the position vector of the point P (x, y, z).



In component form $\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

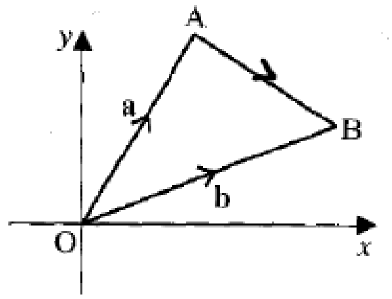
So $\underline{\boldsymbol{p}} = \overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

In the diagram to the right,

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\overrightarrow{AB} = \underline{\boldsymbol{b}} - \underline{\boldsymbol{a}}$$



This is true for all vectors ie $\overrightarrow{ST} = \underline{\boldsymbol{t}} - \underline{\boldsymbol{s}}$

Examples

V-01 A is the point (-12, 4) and B is the point (5, -2).

(a) Write the components of the position vectors **a** and **b**.

(b) Find the components of $\overrightarrow{AB}$

(a) $\underline{a} = \begin{pmatrix} -12 \\ 4 \end{pmatrix}$ $\quad \underline{b} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

(b) $\overrightarrow{AB} = \underline{b} - \underline{a}$

$= \begin{pmatrix} 5 \\ -2 \end{pmatrix} - \begin{pmatrix} -12 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 17 \\ -6 \end{pmatrix}$

V-02 For each pair of points find (i) the components of $\overrightarrow{PQ}$

(ii) $|\overrightarrow{PQ}|$

(a) P (2, 5) Q (4,-2)
(b) P (-2, 4, 5) Q (-3, 0, -2)
(c) P (2, 0, -12) Q (-2, 6, 0)

(a)
$$\overrightarrow{PQ} = \underset{\sim}{q} - \underset{\sim}{p} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \end{pmatrix}$$

$$|\overrightarrow{PQ}| = \sqrt{2^2 + (-7)^2} = \sqrt{53}$$

(b)
$$\overrightarrow{PQ} = \underset{\sim}{q} - \underset{\sim}{p} = \begin{pmatrix} -3 \\ 0 \\ -2 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ -7 \end{pmatrix}$$

$$|\overrightarrow{PQ}| = \sqrt{(-1)^2 + (-4)^2 + (-7)^2} = \sqrt{66}$$

(c)
$$\overrightarrow{PQ} = \underset{\sim}{q} - \underset{\sim}{p} = \begin{pmatrix} -2 \\ 6 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ -12 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \\ 12 \end{pmatrix}$$

$$|\overrightarrow{PQ}| = \sqrt{(-4)^2 + 6^2 + 12^2} = \sqrt{196} = 14$$

Unit Vector

A unit vector is a vector with magnitude equal to 1

To calculate the unit vector of a given vector we divide each component by the magnitude of the vector.

Examples

V-03 Calculate the unit vector of $\underline{\boldsymbol{u}} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$

$$|\underline{u}| = \sqrt{3^2 + 0^2 + 4^2}$$
$$= 5$$

$$\boxed{\begin{array}{l} \text{Unit vector } \underline{u} = \frac{1}{5}\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \\ \underline{\text{OR}} \quad \underline{u} = \begin{pmatrix} 3/5 \\ 0 \\ 4/5 \end{pmatrix} \end{array}}$$

V-04 If $\underline{\boldsymbol{a}} = \begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix}$ and $k\underline{\boldsymbol{a}}$ is a unit vector, calculate the value of k.

$$|\underline{a}| = \sqrt{(-5)^2 + (2)^2 + (4)^2}$$
$$= \sqrt{45}$$

$$\boxed{k = \frac{1}{\sqrt{45}}}$$

Parallel Vectors and Collinearity

If vector $\underline{\boldsymbol{v}} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ then $k\underline{\boldsymbol{v}} = \begin{pmatrix} 2k \\ -1k \\ 3k \end{pmatrix}$ then vector $k\underline{\boldsymbol{v}}$ is parallel to vector $\underline{\boldsymbol{v}}$.

Hence, if $\underline{\boldsymbol{u}} = k\underline{\boldsymbol{v}}$ then $\underline{\boldsymbol{u}}$ is parallel to $\underline{\boldsymbol{v}}$.

Note: if k<0 then the vectors are still parallel but facing in opposite directions.

We know from the straight line chapter that points are collinear if they lie on a straight line.

So if $\overrightarrow{AB} = k\overrightarrow{BC}$ where k is a scalar, then $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$. They share a common point, B, meaning A, B and C **must** be collinear.

Examples

V-05 P is (2, -3, 5), Q is (9, -6, 9) and R is (23, -12, 17). Prove the points are collinear.

$$\overrightarrow{PQ} = \underset{\sim}{q} - \underset{\sim}{p} = \begin{pmatrix} 9 \\ -6 \\ 9 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ 4 \end{pmatrix}$$

$$\overrightarrow{QR} = \underline{r} - \underset{\sim}{q} = \begin{pmatrix} 23 \\ -12 \\ 17 \end{pmatrix} - \begin{pmatrix} 9 \\ -6 \\ 9 \end{pmatrix} = \begin{pmatrix} 14 \\ -6 \\ 8 \end{pmatrix}$$

$\overrightarrow{QR} = 2\overrightarrow{PQ}$ SO VECTORS ARE PARALLEL, SHARE COMMON POINT Q SO POINTS ARE COLLINEAR

V-06 Points E (-1, 4, 8), F (1, 2, 3) and G (5, y, z) are collinear. Find the values of y and z.

$$\overrightarrow{EF} = \underset{\sim}{f} - \underline{e} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -5 \end{pmatrix}$$

$$\overrightarrow{FG} = \underset{\sim}{g} - \underset{\sim}{f} = \begin{pmatrix} 5 \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ y-2 \\ z-3 \end{pmatrix}$$

BECAUSE E, F & G ARE COLLINEAR THEN $\overrightarrow{EF}$ & $\overrightarrow{FG}$ ARE PARALLEL

$$\begin{pmatrix} 4 \\ y-2 \\ z-3 \end{pmatrix} = 2\begin{pmatrix} 2 \\ -2 \\ -5 \end{pmatrix}$$

$4 = 4$

$y - 2 = -4$

$z - 3 = -10$

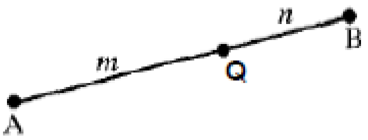
$y - 2 = -4$, $y = -2$ $\qquad$ $z - 3 = -10$, $z = -7$

Splitting a Vector in a Given Ratio

Section formula

If $\underline{\boldsymbol{q}}$ is the position vector of the point Q that divides AB in the ratio of *m:n* then

$$\underline{\boldsymbol{q}} = \left(\frac{n}{m+n}\right)\underline{\boldsymbol{a}} + \left(\frac{m}{n+m}\right)\underline{\boldsymbol{b}}$$



This is called the **section formula**.

You either need to remember this or how to solve these types of questions using an alternative method.

Alternate Methods

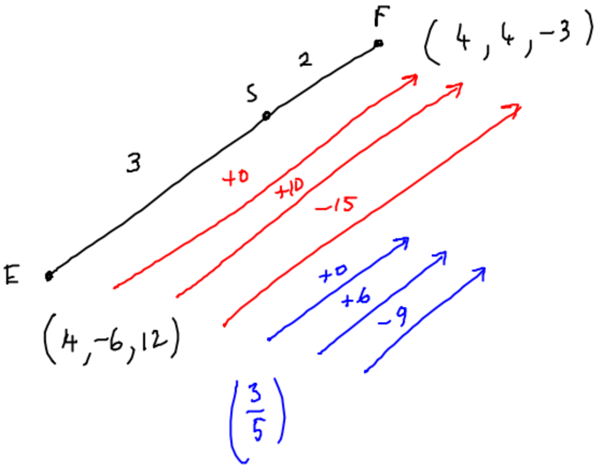
It is possible to find the coordinates of Point Q by an alternative method. Following these steps will result in obtaining the coordinates of Q:

- Find $\overrightarrow{AB}$
- Calculate the fraction of $\overrightarrow{AB}$ based on the ratio that Q splits AB
- Add this to position vector $\underline{\boldsymbol{q}}$
- Change to coordinates

It is also possible to complete these calculations using a diagram.

Examples

V-07 If E is (4, -6, 12), F (4, 4, -3) and S divides EF in the ratio of 3:2, find the coordinates of S.



$\boxed{S\ (4, 0, 3)}$

OR

$$\overrightarrow{EF} = \underset{\sim}{f} - \underset{\sim}{e}$$

$$= \begin{pmatrix} 4 \\ 4 \\ -3 \end{pmatrix} - \begin{pmatrix} 4 \\ -6 \\ 12 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 10 \\ -15 \end{pmatrix}$$

$$\overrightarrow{ES} = \frac{3}{5}\overrightarrow{EF}$$

$$= \frac{3}{5}\begin{pmatrix} 0 \\ 10 \\ 15 \end{pmatrix}$$

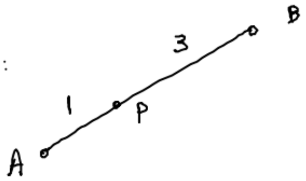
$$= \begin{pmatrix} 0 \\ 6 \\ 9 \end{pmatrix}$$

V-08 A and B have coordinates (3, 2) and (7, 14). If $\frac{AP}{PB} = \frac{1}{3}$, find the coordinates of the point P.

$$\frac{AP}{PB} = \frac{1}{3}$$

$$3AP = PB$$

So:



$$\overrightarrow{AB} = \underline{b} - \underline{a}$$

$$= \begin{pmatrix} 7 \\ 14 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 12 \end{pmatrix}$$

$$\overrightarrow{AP} = \frac{1}{4} \overrightarrow{AB}$$

$$= \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\boxed{P\ (4, 5)}$$

i, j, k Components

All three-dimensional vectors can be written in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, where:

* $\boldsymbol{i}$ is a unit vector in the x direction
* $\boldsymbol{j}$ is a unit vector in the y direction
* $\boldsymbol{k}$ is a unit vector in the z direction

In component form these are written as $i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Examples

V-09 Express each vector in component form

(a) $\underline{\boldsymbol{u}} = 2i + 5j - 3k$

$\underline{u} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$

(b) $\underline{\boldsymbol{v}} = i - 3j + 2k$

$\underline{v} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$

V-10 Express each vector in terms of i, j, and k

(a) $\underline{\boldsymbol{p}} = \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix}$

$\underline{p} = 2i - 6j + 3k$

(b) $\overrightarrow{AB} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$

$\overrightarrow{AB} = 5i - 2k$

V-11 If $\underline{s} = 2i + j - 4k$ and $\underline{t} = -3i + 2j + k$, calculate

(a) $\underline{s} - \underline{t}$ (b) $\underline{s} + \underline{t}$ (c) $2\underline{s} - 3\underline{t}$ (d) $|3\underline{t} + 2\underline{s}|$

(a) $\underline{s} - \underline{t}$

$$= \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} - \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ -1 \\ -5 \end{pmatrix}$$

(b) $\underline{s} + \underline{t}$

$$= \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 3 \\ -3 \end{pmatrix}$$

(c) $2\underline{s} - 3\underline{t}$

$$= 2\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} - 3\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 2 \\ -8 \end{pmatrix} - \begin{pmatrix} -9 \\ 6 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 13 \\ -4 \\ -11 \end{pmatrix}$$

(d) $3\underline{t} + 2\underline{s}$

$$= 3\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} -9 \\ 6 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \\ -8 \end{pmatrix}$$

$$|3\underline{t} + 2\underline{s}| = \sqrt{(-5)^2 + 8^2 + (-5)^2}$$

$$= \sqrt{114}$$

V-12 $\underline{u} = 2i + 4j - 3k$ and $\underline{v} = ai + j - 5k$. If $|\underline{u}| = |\underline{v}|$ calculate a.

$$\underline{u} = \sqrt{(2)^2 + (4)^2 + (-3)^2}$$

$$= \sqrt{29}$$

$$|\underline{v}| = \sqrt{a^2 + (1)^2 + (-5)^2}$$

$$= \sqrt{a^2 + 26}$$

$$|\underline{u}| = |\underline{v}|$$

$$\sqrt{29} = \sqrt{a^2 + 26}$$

$$29 = a^2 + 26$$

$$a^2 = 3$$

$$a = \pm\sqrt{3}$$

$$\boxed{a = \pm\sqrt{3}}$$

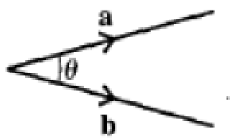
Scalar Product

So far we have added, subtracted and multiplied a vector by a scalar. The scalar product allows us to multiply two vectors together. The scalar product is also called the dot product.

Note: The vectors must be pointing away from each other (and the vertex)

The scalar product is not a vector (it does not have direction)

The scalar product is defined as $\boxed{\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta}$ where θ is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $0 < \theta < 180^o$

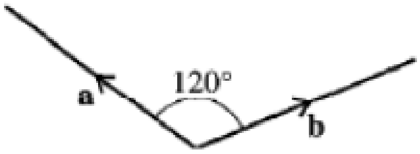


Note: If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular then $\boldsymbol{a}.\boldsymbol{b} = \mathbf{0}$

Examples

V-13 Calculate the scalar product of

(a)



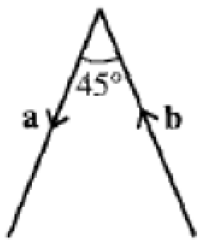
$|\mathbf{a}| = 4,\ |\mathbf{b}| = 6$

$\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$

$= 4 \times 6 \times \cos 120$

$= -12$

(b)



$|\mathbf{a}| = 7,\ |\mathbf{b}| = 8$

$\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$

$= 7 \times 8 \times \cos 135$

$= -28\sqrt{2}$

The scalar product can also be calculated as $\boxed{\boldsymbol{a}.\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3}$

if $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

This is known as the **component form** of the scalar product

Examples

V-14 Calculate the scalar product of

(a) $\underline{\boldsymbol{u}} = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$ $\underline{\boldsymbol{v}} = \begin{pmatrix} -2 \\ 6 \\ 3 \end{pmatrix}$ (b) $\underline{\boldsymbol{s}} = 2i + j - 4k$ $\underline{\boldsymbol{t}} = -3i + 2j + k$

(a) $\underline{u} \cdot \underline{v} = 2(-2) + 0(6) + 5(3)$

$= -4 + 0 + 15$

$= 11$

(b) $\underline{s} \cdot \underline{t} = 2(-3) + 1(2) + (-4)(1)$

$= -6 + 2 - 4$

$= -8$

V-15 A is the point (2, 5, -1), B is (3, -7, 2) and C is (-1, -2, -3). Find

(a) $\overrightarrow{AB}.\overrightarrow{AC}$ (b) $\overrightarrow{BA}.\overrightarrow{BC}$

(a) $\overrightarrow{AB} = \underline{b} - \underline{a}$

$= \begin{pmatrix} 3 \\ -7 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -12 \\ 3 \end{pmatrix}$

$\overrightarrow{AC} = \underline{c} - \underline{a}$

$= \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} -3 \\ -7 \\ -2 \end{pmatrix}$

$\overrightarrow{BC} = \underline{c} - \underline{b}$

$= \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} - \begin{pmatrix} 3 \\ -7 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} -4 \\ 5 \\ -5 \end{pmatrix}$

$\overrightarrow{BA} = \begin{pmatrix} -1 \\ 12 \\ -3 \end{pmatrix}$

(a) $\overrightarrow{AB} \cdot \overrightarrow{AC} = 1(-3) + (-12)(-7) + 3(-2)$

$= -3 + 84 - 6$

$= 75$

(b) $\overrightarrow{BA} \cdot \overrightarrow{BC} = (-1)(-4) + 12(5) + (-3)(-5)$

$= 4 + 60 + 15$

$= 79$

Angle between Two Vectors

We know

$$\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$$

$$\boldsymbol{a}.\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$$

so

$$|\boldsymbol{a}||\boldsymbol{b}|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$$

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\boldsymbol{a}||\boldsymbol{b}|}$$

$$\cos\theta = \frac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

Examples

V-16 Calculate the angle between vectors $\underline{\boldsymbol{s}} = 2i + j - 4k$ and $\underline{\boldsymbol{t}} = -3i + 2j + k$

$$\underline{s}\cdot\underline{t} = 2(-3) + 1(2) + (-4)(1)$$
$$= -6 + 2 - 4$$
$$= -8$$

$$|\underline{s}| = \sqrt{(2)^2 + (1)^2 + (-4)^2}$$
$$= \sqrt{21}$$

$$|\underline{t}| = \sqrt{(-3)^2 + (2)^2 + (1)^2}$$
$$= \sqrt{14}$$

$$\cos\theta = \frac{\underline{s}\cdot\underline{t}}{|\underline{s}||\underline{t}|}$$
$$= \frac{-8}{\sqrt{21}\sqrt{14}}$$
$$\theta = \cos^{-1}\left(\frac{-8}{\sqrt{21}\sqrt{14}}\right)$$
$$= 117.8°$$

V-17 If A is the point (2, 5, -1), and B is (3, -7, 2). Calculate the size of angle AOB.

$$\vec{OA} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \qquad \vec{OB} = \begin{pmatrix} 3 \\ -7 \\ 2 \end{pmatrix}$$

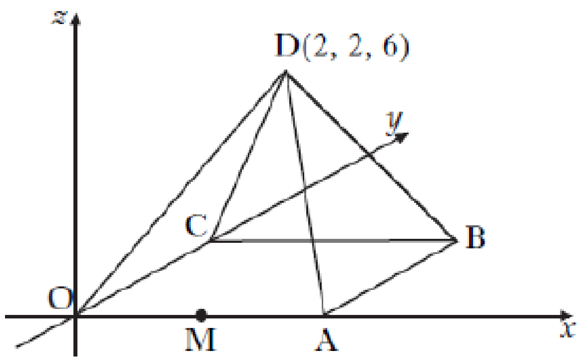
$$\vec{OA}\cdot\vec{OB} = 2(3) + 5(-7) + 1(2)$$
$$= 6 - 35 + 2$$
$$= -27$$

$$|\vec{OA}| = \sqrt{(2)^2 + (5)^2 + (1)^2}$$
$$= \sqrt{30}$$

$$|\vec{OB}| = \sqrt{(3)^2 + (-7)^2 + (2)^2}$$
$$= \sqrt{62}$$

$$\cos AOB = \frac{\vec{OA}\cdot\vec{OB}}{|\vec{OA}||\vec{OB}|}$$
$$= \frac{-27}{\sqrt{30}\sqrt{62}}$$
$$AOB = \cos^{-1}\left(\frac{-27}{\sqrt{30}\sqrt{62}}\right)$$
$$= 128.8°$$

V-18 D,OABC is a square based pyramid as shown in the diagram below.



O is the origin, D is the point $(2, 2, 6)$ and OA $= 4$ units.

M is the mid-point of OA.

(*a*) State the coordinates of B. 1

(*b*) Express $\overrightarrow{DB}$ and $\overrightarrow{DM}$ in component form. 3

(*c*) Find the size of angle BDM. 5

(a) B $(4, 4, 0)$

(b) D $(2, 2, 6)$ m $(2, 0, 0)$

$\overrightarrow{DB} = \underline{b} - \underline{d}$

$= \begin{pmatrix}4\\4\\0\end{pmatrix} - \begin{pmatrix}2\\2\\6\end{pmatrix}$

$= \begin{pmatrix}2\\2\\-6\end{pmatrix}$

$\overrightarrow{Dm} = \underset{\sim}{m} - \underset{\sim}{d}$

$= \begin{pmatrix}2\\0\\0\end{pmatrix} - \begin{pmatrix}2\\2\\6\end{pmatrix}$

$= \begin{pmatrix}0\\-2\\-6\end{pmatrix}$

(c) $\overrightarrow{DB} \cdot \overrightarrow{Dm} = 2(0) + 2(-2) + (-6)(-6)$

$= 0 - 4 + 36$

$= 32$

$|\overrightarrow{DB}| = \sqrt{(2)^2 + (2)^2 + (-6)^2}$

$= \sqrt{44}$

$|\overrightarrow{Dm}| = \sqrt{(0)^2 + (-2)^2 + (-6)^2}$

$= \sqrt{40}$

$\cos BDm = \dfrac{\overrightarrow{DB} \cdot \overrightarrow{Dm}}{|\overrightarrow{DB}||\overrightarrow{Dm}|}$

$= \dfrac{32}{\sqrt{44}\sqrt{40}}$

$BDm = \cos^{-1}\left(\dfrac{32}{\sqrt{44}\sqrt{40}}\right)$

$= 40.3^\circ$

Applications of the Scalar Product

We know that $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ and $\cos 90 = 0$, so if $\boldsymbol{a}.\boldsymbol{b} = 0$ then vectors are perpendicular.

If $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are vectors then $\quad \boldsymbol{a}.(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{c}$

Examples

V-19 If $\underline{s} = 2i + j + ak$ and $\underline{t} = -3i + 2j + 2k$ are perpendicular, calculate a.

IF $\underline{s}$ AND $\underline{t}$ ARE PERP $\underline{s}.\underline{t} = 0$

$$\underline{s}.\underline{t} = 2(-3) + 1(2) + a(2)$$
$$= -6 + 2 + 2a$$
$$= 2a - 4$$

$$2a - 4 = 0$$
$$2a = 4$$
$$a = 2$$

V-20 Calculate $\boldsymbol{u}.(\boldsymbol{v}+\boldsymbol{w})$ when $\underline{\boldsymbol{u}} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \quad \underline{\boldsymbol{v}} = \begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix} \quad \underline{\boldsymbol{w}} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

$$\underline{u}.(\underline{v} + \underline{w})$$
$$= \underline{u}.\underline{v} + \underline{u}.\underline{w}$$
$$= 16 + (-10)$$
$$= 6$$

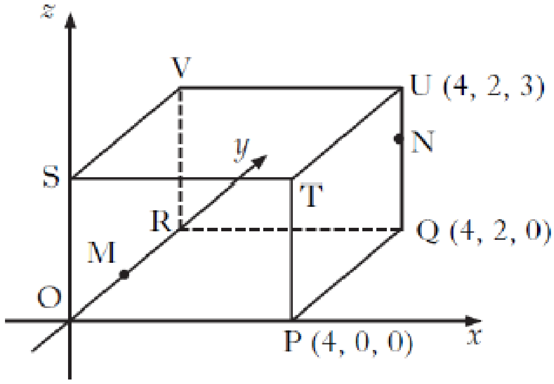
$$\underline{u}.\underline{v} = 2(-1) + 5(4) + 1(-2)$$
$$= -2 + 20 - 2$$
$$= 16$$
$$\underline{u}.\underline{w} = 2(2) + 5(-3) + 1(1)$$
$$= 4 - 15 + 1$$
$$= -10$$

V-21 The diagram shows a cuboid OPQR,STUV relative to the coordinate axes.

P is the point $(4, 0, 0)$, Q is $(4, 2, 0)$ and U is $(4, 2, 3)$.

M is the midpoint of OR.

N is the point on UQ such that $UN = \frac{1}{3}UQ$.



(a) State the coordinates of M and N. **2**

(b) Express the vectors $\overrightarrow{VM}$ and $\overrightarrow{VN}$ in component form. **2**

(c) Calculate the size of angle MVN. **5**

(a) $M(0, 1, 0)$ $\quad N(4, 2, 2)$

(b) $\overrightarrow{VM} = \underline{m} - \underline{v}$ $\quad \overrightarrow{VN} = \underline{n} - \underline{v}$

$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$ $\quad = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix}$ $\quad = \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix}$

(c)

$\overrightarrow{VM} \cdot \overrightarrow{VN} = 0(4) + (-3)(-2) + (-3)(-1)$

$= 0 + 6 + 3$

$= 9$

$|\overrightarrow{VM}| = \sqrt{(0)^2 + (-3)^2 + (-3)^2}$

$= \sqrt{18}$

$|\overrightarrow{VN}| = \sqrt{(4)^2 + (-2)^2 + (-1)^2}$

$= \sqrt{21}$

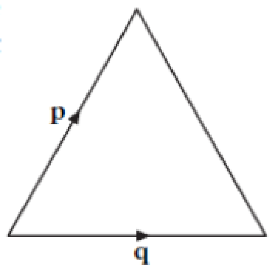
$\cos MVN = \dfrac{\overrightarrow{VM} \cdot \overrightarrow{VN}}{|\overrightarrow{VM}||\overrightarrow{VN}|}$

$= \dfrac{9}{\sqrt{18}\sqrt{21}}$

$MVN = \cos^{-1}\left(\dfrac{9}{\sqrt{18}\sqrt{21}}\right)$

$= 62.4°$

V-22 An equilateral triangle of side 5 units is shown. The vectors p and q are as represented in the diagram. What is the value of $p.q$?



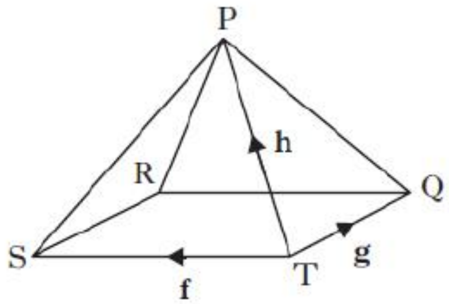
$$p \cdot q = |p||q| \cos\theta$$
$$= 5 \times 5 \times \cos 60$$
$$= \frac{25}{2}$$

Vector Journeys

It is possible to describe a vector as a combination of other vectors. Doing this "creates" a journey.

Examples

V-23 The diagram shows a square-based pyramid P,QRST. $\overrightarrow{TS}$, $\overrightarrow{TQ}$ and $\overrightarrow{TP}$ represent f, g and h respectively.

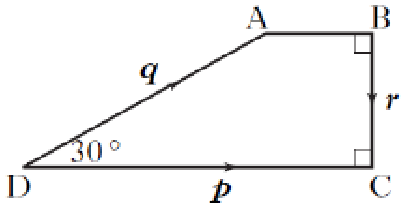


Express $\overrightarrow{RP}$ in terms of f, g and h.

$\overrightarrow{RP} = -f - g + h$

$\overrightarrow{RQ} = -f \quad \overrightarrow{QT} = -g \quad \overrightarrow{TP} = h$

V-24 Vectors $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$ are represented on the diagram shown where angle ADC = 30°.



It is also given that $|\boldsymbol{p}| = 4$ and $|\boldsymbol{q}| = 3$.

(a) Evaluate $\boldsymbol{p}.(\boldsymbol{q}+\boldsymbol{r})$ and $\boldsymbol{r}.(\boldsymbol{p}-\boldsymbol{q})$. 6

(b) Find $|\boldsymbol{q}+\boldsymbol{r}|$ and $|\boldsymbol{p}-\boldsymbol{q}|$. 4

(a) $\underline{p}\cdot(\underline{q}+\underline{r})$

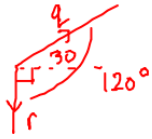
$= \underline{p}\cdot\underline{q} + \underline{p}\cdot\underline{r}$

$= 4\times3\times\cos30° + 4\times|r|\times\cos90°$

$= 6\sqrt{3} + 0$

$= 6\sqrt{3}$

$\underline{r}\cdot(\underline{p}-\underline{q})$



$= \underline{r}\cdot\underline{p} - \underline{r}\cdot\underline{q}$

$= |r|\times4\times\cos90° - |r|\times3\times\cos120°$

$= 0 - \left(-\frac{3}{2}|r|\right)$

$= 0 + \frac{9}{4}$

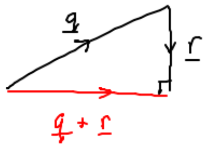
$= \frac{9}{4}$

$\sin 30 = \frac{|r|}{3}$

$|r| = 3\sin30$

$= \frac{3}{2}$

(b) $|\underline{q}+\underline{r}|$



PYTHAGORAS

$|\underline{q}+\underline{r}|^2 = \sqrt{|\underline{q}|^2 - |r|^2}$

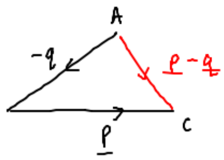
$= \sqrt{3^2 - \left(\frac{3}{2}\right)^2}$

$= \sqrt{9 - \frac{9}{4}}$

$= \sqrt{\frac{27}{4}}$

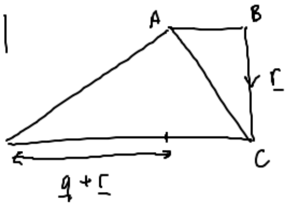
$= \frac{\sqrt{27}}{2} = \frac{3\sqrt{3}}{2}$

$|\underline{p}-\underline{q}| = |-\underline{q}+\underline{p}|$



$\vec{AB} = 4 - |\underline{q}+\underline{r}|$

$= 4 - \frac{3\sqrt{3}}{2}$



$|\underline{p}-\underline{q}|^2 = \sqrt{\left(4-\frac{3\sqrt{3}}{2}\right)^2 + \left(\frac{3}{2}\right)^2}$

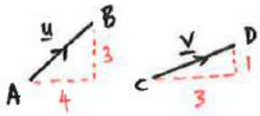
$= 2.05$

Summary

Vectors

RECAP FROM NAT 5:

A vector has direction and magnitude



$\underline{u} = \overrightarrow{AB} = \begin{pmatrix}4\\3\end{pmatrix}$ $\quad \underline{v} = \overrightarrow{CD} = \begin{pmatrix}3\\1\end{pmatrix}$

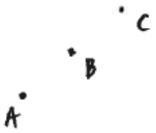
$$2\underline{u} + 3\underline{v} = 2\begin{pmatrix}4\\3\end{pmatrix} + 3\begin{pmatrix}3\\1\end{pmatrix}$$
$$= \begin{pmatrix}8\\6\end{pmatrix} + \begin{pmatrix}9\\3\end{pmatrix}$$
$$= \begin{pmatrix}17\\9\end{pmatrix}$$

magnitude is length:

$$|\underline{u}| = \sqrt{4^2+3^2}$$
$$= \sqrt{25}$$
$$= 5 \text{ units}$$

RULES WORK WITH 3D VECTORS TOO!

COLLINEARITY: Similar to straight line topic



1. Find $\overrightarrow{AB}$ & $\overrightarrow{BC}$
2. Check parallel
3. State "parallel and common point B so collinear"

PARALLEL VECTORS: parallel if one vector is a multiple of the other

i, j, k:

$\underline{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $\quad \underline{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $\quad \underline{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$\underline{i}$ is movement on x-axis
$\underline{j}$ is movement on y-axis
$\underline{k}$ is movement on z-axis

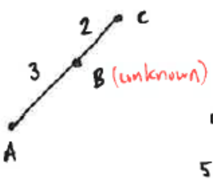
SCALAR PRODUCT (DOT PRODUCT): two ways to calculate, both are given in formulae sheet

$\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$ ← angle between vectors $\underline{a}$ and $\underline{b}$

OR

$\underline{a}.\underline{b} = a_1b_1 + a_2b_2 + a_3b_3$

FINDING COORDINATES OF A POINT ON A LINE:



$\overrightarrow{AB} = \frac{3}{5}\overrightarrow{AC}$ ← 5 "parts" in whole line

$5\overrightarrow{AB} = 3\overrightarrow{AC}$

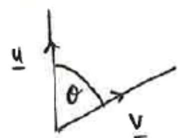
$5(\underline{b}-\underline{a}) = 3(\underline{c}-\underline{a})$

then solve!

REMEMBER: YOU MUST WRITE COORDINATES HORIZONTALLY!!!

☆ More of a challenge?
$\underline{a}.(\underline{b}+\underline{c}) = \underline{a}.\underline{b} + \underline{a}.\underline{c}$

ANGLE BETWEEN VECTORS:



$$\cos\theta = \frac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|}$$

not given in an exam, rearrange scalar product.

Vectors must be pointing away from each other!

For perpendicular vectors:

$\underline{a}.\underline{b} = 0$