HIGHER MATHS

Recurrence Relations

Notes with Examples

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What is a Recurrence Relation?

A recurrence relation describes a sequence in which each term is a function of the previous term.

Terms in a recurrence relation are labelled $u_0, u_1, u_2, u_3 \ldots$ where u_0 is the starting value, u_1 is the first term, u_2 is the second term and so on.

A basic recurrence relation can be written as

$$u_{n+1} = au_n$$

where u_n is the n^{th} term in the sequence and a is a constant.

Examples

R-01 £2000 is invested at in interest rate of 4% per annum.

(a) Find a recurrence relation for the value of the investment
(b) Calculate the value of the investment after 3 years

(a) $u_{n+1} = 1.04\,u_n$ ← 100% + 4% = 1.04

(b) $u_1 = 1.04 \times 2000$
$= 2080$
$u_2 = 1.04 \times 2080$
$= 2163.20$
$u_3 = 1.04 \times 2163.20$
$= 2249.728$

VALUE AFTER 3 YEARS
£2249.72

R-02 A city council wants to reduce the level of air pollution in the city to less than 1.0mg/m^3. The current level is 1.4mg/m^3 and the council plan to reduce this level by 5% per annum.

(a) Find a recurrence relation for the level of pollution
(b) How long will it take for the council to achieve their target?

(a) $u_{n+1} = 0.95\,u_n$ ← 100% − 5% = 0.95

(b) $u_1 = 0.95 \times 1.4$
≈ 1.33
$u_2 = 1.2635$
$u_3 = 1.200$
$u_4 = 1.140$
$u_5 = 1.08$

DO THIS QUICKLY ON A CALCULATOR
1.4 $\boxed{=}$
0.95 × $\boxed{\text{ANS}}$
THEN KEEP HITTING EQUALS BUTTON

IT WILL TAKE 5 YEARS

Linear Recurrence Relations

A linear recurrence relation has the form:

$$u_{n+1} = au_n + b$$

Examples

R-03 For these recurrence relations find u_4.

(a) $u_{n+1} = 3u_n - 50, \quad u_0 = 100$

(b) $u_{n+1} = -0.5u_n + 2, \quad u_0 = 6$

(a) $u_1 = 3 \times 100 - 50$
$= 250$
$u_2 = 700$
$u_3 = 2050$
$u_4 = 6100$

(b) $u_1 = -0.5 \times 6 + 2$
$= -1$
$u_2 = 2.5$
$u_3 = 0.75$
$u_4 = 1.625$

R-04 A company has applied to dump 200 litres of processing waste per week into a loch. It is estimated that the natural action of the sea will remove 30% of waste per week.

(a) Find a recurrence relation for the level of processing waste in the loch at the end of each week.

(b) What will the level of chemical waste be after
(i) 2 weeks (ii) 4 weeks (iii) 6 weeks

(a) $U_{n+1} = 0.7\,U_n + 200$ 30% REMOVED = 70% REMAINS

(b) $U_1 = 0.7 \times 0 + 200$
$= 200$
$U_2 = 340$ $\boxed{\text{2 WEEKS} = 340\text{ L}}$
$U_3 = 438$
$U_4 = 506.6$ $\boxed{\text{4 WEEKS} = 506.6\text{ L}}$
$U_5 = 554.62$
$U_6 = 588.234$ $\boxed{\text{6 WEEKS} = 588.234\text{ L}}$

Limit of a Sequence

A recurrence relation can be said to be either convergent or divergent. If a recurrence relation finally settles down to one value over a period of time, it is said to be convergent and therefore have a limit.

For a recurrence relation to be convergent then $-1 < a < 1$.

If u_n tends to a limit the limit can be calculated by $L = \frac{b}{1-a}$

Examples

R-05 A company has applied to dump 200 litres of processing waste per week into a loch. It is estimated that the natural action of the sea will remove 30% of waste per week. To be given permission, the loch must have no more than 600 litres of processing waste in the long term.

(a) Find a recurrence relation for the level of processing waste in the loch at the end of each week

(b) Should the company be allowed to dump the processing waste into the Loch?

(c) The company agree to reduce the amount of processing waste to 170 litres per week. Should they now be given permission?

(a) $u_{n+1} = 0.7u_n + 200$

(b) LIMIT EXISTS AS $-1 < 0.7 < 1$

$L = 0.7L + 200$

$L - 0.7L = 200$

$0.3L = 200$

$L = 666.67\ l$

OR USE $L = \frac{b}{1-a}$

NO THEY SHOULD NOT BE ALLOWED AS THE LIMIT IS $666.67 > 600$

(c)

$L = 0.7L + 170$

$L - 0.7L = 170$

$0.3L = 170$

$L = 566.67$

YES THEY SHOULD BE ALLOWED AS THE LIMIT IS $566.67 < 600$

R-06 A frog and a toad fall to the bottom of a well that is 50 feet deep.

Each day, the frog climbs 32 feet and then rests overnight. During the night, it slides down $\frac{2}{3}$ of its height above the floor of the well.

The toad climbs 13 feet each day before resting.

Overnight, it slides down $\frac{1}{4}$ of its height above the floor of the well.

Their progress can be modelled by the recurrence relations:

- $f_{n+1} = \frac{1}{3} f_n + 32, \qquad f_1 = 32$
- $t_{n+1} = \frac{3}{4} t_n + 13, \qquad t_1 = 13$

where f_n and t_n are the heights reached by the frog and the toad at the end of the nth day after falling in.

(a) Calculate t_2, the height of the toad at the end of the second day. **1**

(b) Determine whether or not either of them will eventually escape from the well. **5**

(a) $t_2 = \frac{3}{4} \times 13 + 13$

$= \frac{39}{4} + \frac{52}{4}$

$= \frac{91}{4}$ ft $\left(22\frac{3}{4}\text{ ft}\right)$

(b) FROG

LIMIT EXISTS AS $-1 < \frac{1}{3} < 1$

$L = \frac{1}{3}L + 32$

$L - \frac{1}{3}L = 32$

$\frac{2}{3}L = 32$

$2L = 96$

$L = 48$

$\boxed{\text{FROG DOES NOT ESCAPE}}$

TOAD

LIMIT EXISTS AS $-1 < \frac{3}{4} < 1$

$L = \frac{3}{4}L + 13$

$L - \frac{3}{4}L = 13$

$\frac{1}{4}L = 13$

$L = 52$

$\boxed{\text{TOAD DOES ESCAPE}}$

R-07 A recurrence relation is given by $u_{n+1} = ku_n + 5k$. Given the limit of the sequence is 40, find the value of k.

$L = kL + 5k \qquad L = 40$

$40 = 40k + 5k$

$40 = 45k$

$k = \frac{40}{45}$

$k = \frac{8}{9}$

Stating a Recurrence Relation for a Given Sequence

If given a sequence of terms, it is possible to find the recurrence relation that connects them. To do this we used simultaneous equations.

Examples

R-08 A recurrence relation is defined by $u_{n+1} = au_n + b$. Find the values of a and b for the following sequences.

(a) $u_0 = 100, \quad u_1 = 60, \quad u_2 = 52$

(b) $u_5 = 12.5, \quad u_6 = 16.75, \quad u_7 = 23.125$

(a)
$$u_{n+1} = au_n + b$$
$$60 = 100a + b \quad — ①$$
$$52 = 60a + b \quad — ②$$
① - ②
$$8 = 40a$$
$$a = 0.2$$
SUB $a = 0.2$ INTO ①
$$60 = 0.2 \times 100 + b$$
$$60 = 20 + b$$
$$b = 40$$

(b)
$$u_{n+1} = au_n + b$$
$$16.75 = 12.5a + b \quad — ①$$
$$23.125 = 16.75a + b \quad — ②$$
① - ②
$$-6.375 = -4.5a$$
$$a = 1.5$$
SUB $a = 1.5$ INTO ①
$$16.75 = 12.5 \times 1.5 + b$$
$$16.75 = 18.75 + b$$
$$b = -2$$

Linked Recurrence Relations

Linked recurrence relations require us to use two different formulae to solve one problem.

Examples

R-09 A car hire company has depots in Glasgow and Perth. Between them they have 200 cars.

It can be shown that of the cars hired in Perth, 80% are returned there with the remaining cars returned to Glasgow.

Of the cars hired in Glasgow 60% are returned there while 40% are returned to Perth.

How many cars should be stored at each depot?

P_n = NUMBER OF CARS AT PERTH

G_n = NUMBER OF CARS AT GLASGOW

$P_n + G_n = 200$

$G_n = 200 - P_n$

$$P_{n+1} = \underbrace{0.8P_n}_{\text{80\% PERTH}} + \underbrace{0.4G_n}_{\text{40\% GLASGOW}}$$

$$= 0.8P_n + 0.4(200 - P_n)$$

$$= 0.8P_n + 80 - 0.4P_n$$

$$= 0.4P_n + 80$$

LIMIT EXISTS AS $-1 < 0.4 < 1$

$$L = 0.4L + 80$$

$$L - 0.4L = 80$$

$$0.6L = 80$$

$$L = 133 \text{ CARS}$$

$\boxed{\text{133 CARS SHOULD BE STORED AT PERTH, 67 CARS AT GLASGOW}}$

Summary

RECURRENCE RELATIONS

$$U_{n+1} = aU_n + b$$

a sequence where each term is based on the previous term.

a is usually a percentage in decimal form

Does the sequence have a limit?

★ limit exists if $-1 < a < 1$ Remember to state this!

★ to find limit either: set U_{n+1} and u_n to L and solve for L.

or $L = \frac{b}{1-a}$

★ You may be given three consecutive terms u_1, u_2 & u_3 and have to calculate a & b through simultaneous equations.

EXAMPLE: A scientist is studying a large flock of birds. Every minute 20% fly away but 20 new birds join. The initial size of the flock was 300 birds.

(a) write a recurrence relation to model the situation

(b) what can you say about the size of the flock in the long term?

(a) 20% fly away = 80% remain.

$$u_{n+1} = 0.8u_n + 20$$ ← number of "new" birds

(b) Size of flock will tend to a limit $-1 < 0.8 < 1$

$$L = 0.8L + 20$$

$$L - 0.8L = 20$$

$$0.2L = 20$$

$$L = \frac{20}{0.2}$$ ← can you do this without a calculator?

$$L = 100$$