HIGHER MATHS

Quadratics & Polynomials

Notes with Examples

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Completing the Square

Any quadratic function can be written in completed square form $y = (x - a)^2 + b$.

If there is no coefficient in front of x^2 then we complete following the rules set out at National 5. If there is a coefficient we must remove that as a common factor first before completing the square.

Even x

$$\begin{aligned} & x^2 + 8x + 9 \\ = \; & (x+4)^2 + 9 - 4^2 \\ = \; & (x+4)^2 - 7 \end{aligned}$$

Odd x

$$\begin{aligned} & x^2 + 5x - 3 \\ = \; & \left(x + \frac{5}{2}\right)^2 - 3 - \left(\frac{5}{2}\right)^2 \\ = \; & \left(x + \frac{5}{2}\right)^2 - \frac{12}{4} - \frac{25}{4} \\ = \; & \left(x + \frac{5}{2}\right)^2 - \frac{37}{4} \end{aligned}$$

x^2 coefficient

$$\begin{aligned} & 2x^2 + 12x + 10 \\ = \; & 2[x^2 + 6x] + 10 \\ = \; & 2[(x+3)^2 - 3^2] + 10 \\ = \; & 2[(x+3)^2 - 9] + 10 \\ = \; & 2(x+3)^2 - 18 + 10 \\ = \; & 2(x+3)^2 - 8 \end{aligned}$$

Examples

Write the following expressions in completed square form:

QP-01 (a) $x^2 + 6x + 1$ (b) $x^2 - 2x + 11$

(a) $x^2 + 6x + 1$

$= (x+3)^2 + 1 - 9$

$= (x+3)^2 - 8$

(b) $x^2 - 2x + 11$

$= (x-1)^2 + 11 - 1$

$= (x-1)^2 + 10$

QP-02 (a) $x^2 + x + 3$ (b) $x^2 - 5x - 1$

(a) $x^2 + x + 3$

$= \left(x + \frac{1}{2}\right)^2 + 3 - \frac{1}{4}$

$= \left(x + \frac{1}{2}\right)^2 + \frac{11}{4}$

(b) $x^2 - 5x - 1$

$= \left(x - \frac{5}{2}\right)^2 - 1 - \frac{25}{4}$

$= \left(x - \frac{5}{2}\right)^2 - \frac{29}{4}$

QP-03 (a) $2x^2 + 8x + 3$ (b) $3x^2 - 6x + 1$ (c) $6 - x - 2x^2$

(a) $2x^2 + 8x + 3$

$= 2\left[x^2 + 4x\right] + 3$

$= 2\left[(x+2)^2 - 4\right] + 3$

$= 2(x+2)^2 - 8 + 3$

$= 2(x+2)^2 - 5$

(b) $3x^2 - 6x + 1$

$= 3\left[x^2 - 2x\right] + 1$

$= 3\left[(x-1)^2 - 1\right] + 1$

$= 3(x-1)^2 - 3 + 1$

$= 3(x-1)^2 - 2$

(c) $6 - x - 2x^2$

$= -2x^2 - x + 6$

$= -2\left[x^2 + \frac{1}{2}x\right] + 6$

$= -2\left[\left(x + \frac{1}{4}\right)^2 - \frac{1}{16}\right] + 6$

$= -2\left(x + \frac{1}{4}\right)^2 + \frac{1}{8} + 6$

$= -2\left(x + \frac{1}{4}\right)^2 + \frac{49}{8}$

Sketching Quadratics

The graph of a quadratic function $y = ax^2 + bx + c$ is a parabola.

If $a > 0$ the graph has a minimum turning point.

If $a < 0$ the graph has a maximum turning point.

To sketch a graph we need to know:

* Shape of the parabola
* Where it cuts the x-axis (roots)
* Where it cuts the y-axis
* Axis of symmetry (halfway between roots)
* Coordinates of the turning point

If we have a function in completed square form sketching, the graph is straight forward.

For $y = (x - a)^2 + b$

* Turning point is (a, b)
* Axis of symmetry is $x = a$

Examples

QP-04 Sketch: (a) $y = (x + 3)(x + 5)$

(b) $y = x^2 + 4x - 12$

(a) $y = x^2 + 8x + 15$

CUTS X AXIS WHEN $y = 0$

$(x+3)(x+5) = 0$

$x = -3 \quad x = -5$

CUTS Y AXIS WHEN $x = 0$

$y = (0)^2 + 8(0) + 15$

$= 15$

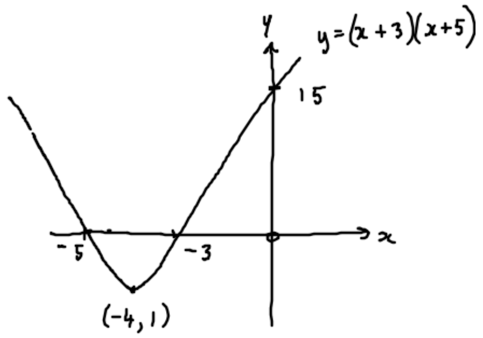
AXIS OF SYMMETRY

$x = \frac{-3+(-5)}{2} = -4$

TURNING POINT

$y = (-4)^2 + 8(-4) + 15$

$= -1$



(b) $y = x^2 + 4x - 12$

$= (x+6)(x-2)$

CUTS X AXIS WHEN $y = 0$

$(x+6)(x-2) = 0$

$x = -6 \quad x = 2$

CUTS Y AXIS WHEN $x = 0$

$y = (0)^2 + 4(0) - 12$

$= -12$

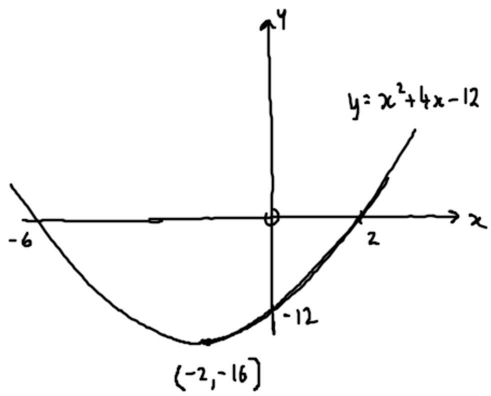
AXIS OF SYMMETRY

$x = \frac{-6+2}{2} = -2$

TURNING POINT

$y = (-2)^2 + 4(-2) - 12$

$= -16$



QP-05 Using completed square form, find the equation of the axis of symmetry, coordinates of the turning point and sketch the graph of:

(a) $y = 2x^2 - 8x + 9$

(b) $y = 7 + 6x - x^2$

(a)
$$
\begin{aligned}
y &= 2x^2 - 8x + 9 \\
&= 2\left[x^2 - 4x\right] + 9 \\
&= 2\left[(x-2)^2 - 4\right] + 9 \\
&= 2(x-2)^2 - 8 + 9 \\
&= 2(x-2)^2 + 1
\end{aligned}
$$

AXIS OF SYMMETRY

$x - 2 = 0$

$x = 2$

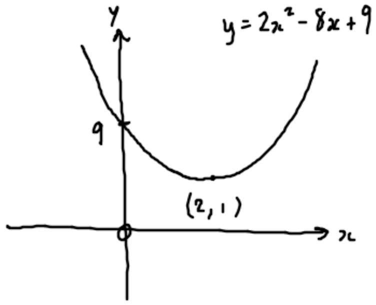
TURNING POINT

$(2, 1)$

CUTS Y AXIS WHEN $X=0$

$Y = 2(0)^2 - 8(0) + 9$

$= 9$



(b)
$$
\begin{aligned}
y &= 7 + 6x - x^2 \\
&= -x^2 + 6x + 7 \\
&= -\left[x^2 - 6x\right] + 7 \\
&= -\left[(x-3)^2 - 9\right] + 7 \\
&= -(x-3)^2 + 9 + 7 \\
&= -(x-3)^2 + 16
\end{aligned}
$$

AXIS OF SYMMETRY

$x - 3 = 0$

$x = 3$

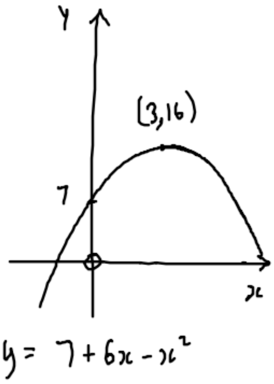
TURNING POINT

$(3, 16)$

CUTS Y AXIS WHEN $X=0$

$y = 7 + 6(0) - (0)^2$

$= 7$



Finding the Equation of a Quadratic from its Graph

To find the equation of a quadratic function from its graph we need to know:

- * The roots (where it cuts x-axis)
- * A single point on the curve

The general form of a quadratic in this case will look like

$$y = k(x - a)(x + b)$$

Examples

QP-06 Find the equations of the following parabolas

(a)

$y = k(x-4)(x+1)$

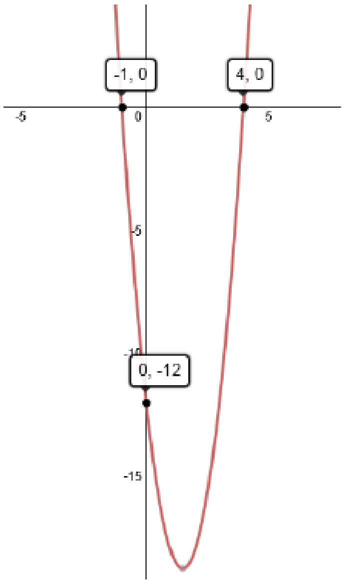
Sub $x=0$ $y=-12$

$-12 = k(-4)(1)$

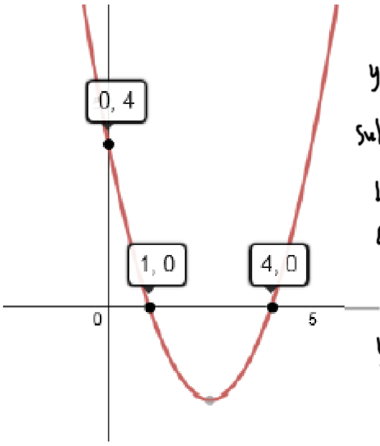
$-12 = -4k$

$k=3$

$y = 3(x-4)(x+1)$



(b)



$y = k(x-1)(x-4)$

Sub $x=0$ $y=4$

$4 = k(-1)(-4)$

$4 = 4k$

$k=1$

$y = (x-1)(x-4)$

(c)

$y = k(x+2)(x-4)$

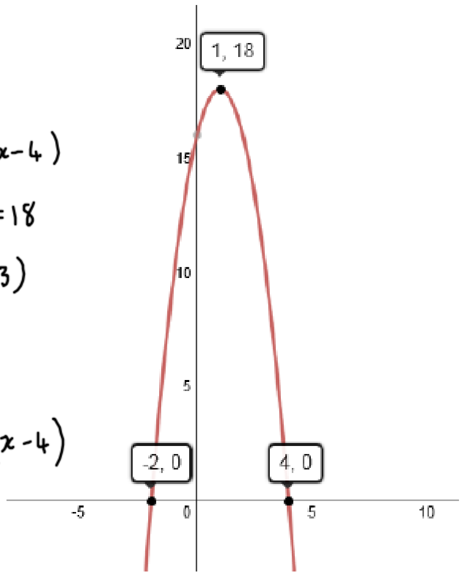
Sub $x=1$ $y=18$

$18 = k(3)(-3)$

$18 = -9k$

$k=-2$

$y = -2(x+2)(x-4)$



(d)

$y = k(x+1)(x-7)$

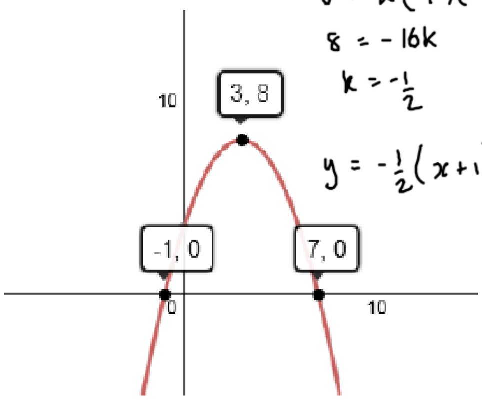
Sub $x=3$ $y=8$

$8 = k(4)(-4)$

$8 = -16k$

$k = -\frac{1}{2}$

$y = -\frac{1}{2}(x+1)(x-7)$



Solving Quadratic Inequalities

To solve a quadratic inequation we have to:

- **Equate** to zero
- Factorise
- Sketch the graph
- Write the solution

When $ax^2 + bx + c > 0$ the solution is when the graph is above the x-axis.

When $ax^2 + bx + c < 0$ the solution is when the graph is below the x-axis.

Examples

QP-07 Solve

(a) $x^2 + 4x - 12 > 0$ (b) $x^2 - 4x < 0$

(c) $7 + 6x - x^2 < 0$ (d) $5x - 10x^2 > 0$

(a) $x^2 + 4x - 12 > 0$

$(x + 6)(x - 2) > 0$

ROOTS : $x = -6$, $x = 2$

SHAPE : $+x^2$ MIN TP

$x < -6$ $x > 2$

(b) $x^2 - 4x < 0$

$x(x - 4) < 0$

ROOTS : $x = 0$ $x = 4$

SHAPE : $+x^2$ MIN TP

$0 < x < 4$

(c) $7 + 6x - x^2 < 0$

$(7 - x)(1 + x) < 0$

ROOTS : $x = -1$ $x = 7$

SHAPE : $-x^2$ MAX TP

$x < -1$ $x > 7$

(d) $5x - 10x^2 > 0$

$5x(1 - 2x) > 0$

ROOTS : $x = 0$ $x = \frac{1}{2}$

SHAPE : $-x^2$ MAX TP

$0 < x < \frac{1}{2}$

Tangents to Parabolas

To determine whether a straight line cuts, touches or does not touch a curve we substitute the equation of the straight line into that of the curve. That will result in a quadratic equation, then we use the discriminant to determine the number of points of intersection.

* If $b^2 - 4ac > 0$ there are two points of intersection
* If $b^2 - 4ac = 0$ there is one point of contact - **tangent**
* If $b^2 - 4ac < 0$ there are no points of intersection

Examples

QP-08 State the number of points of intersection between the curve $y = x^2 + 3x + 2$ and

(a) $y = x + 3$ (b) $y = x + 2$ (c) $y = x + 1$ (d) $y = x - 1$

(a) $x^2 + 3x + 2 = x + 3$
$x^2 + 2x - 1 = 0$
$b^2 - 4ac$
$= 2^2 - (4 \times 1 \times (-1))$
$= 4 - (-4)$
$= 8$
$b^2 - 4ac > 0$ So 2 POINTS OF INTERSECTION

(b) $x^2 + 3x + 2 = x + 2$
$x^2 + 2x = 0$
$b^2 - 4ac$
$= 2^2 - (4 \times 1 \times 0)$
$= 4 - 0$
$= 4$
$b^2 - 4ac > 0$ So 2 POINTS OF INTERSECTION

(c) $x^2 + 3x + 2 = x + 1$
$x^2 + 2x + 1 = 0$
$b^2 - 4ac$
$= 2^2 - (4 \times 1 \times 1)$
$= 4 - 4$
$= 0$
$b^2 - 4ac = 0$ So 1 POINT OF CONTACT

(d) $x^2 + 3x + 2 = x - 1$
$x^2 + 2x + 3 = 0$
$b^2 - 4ac$
$= 2^2 - (4 \times 1 \times 3)$
$= 4 - 12$
$= -8$
$b^2 - 4ac < 0$ So NO POINTS OF INTERSECTION

QP-09 Prove that the line $y = 2x - 1$ is a tangent to the parabola $y = x^2$ and find the point of intersection.

$x^2 = 2x - 1$
$x^2 - 2x + 1 = 0$
$b^2 - 4ac$
$= (-2)^2 - (4 \times 1 \times 1)$
$= 4 - 4$
$= 0$
$b^2 - 4ac = 0$ So 1 POINT OF CONTACT (TANGENT)

$x^2 - 2x + 1 = 0$
$(x - 1)(x - 1) = 0$
$x = 1$
$y = (1)^2$
$= 1$

POC $(1, 1)$

QP-10 Find the equation of the tangent to $y = x^2 + 1$ that has a gradient of 2.

$y = x^2 + 1$ $\quad$ $y = mx + c$

FOR TANGENT $b^2 - 4ac = 0$

$x^2 + 1 = 2x + c$

$x^2 - 2x + 1 - c = 0$

$(-2)^2 - (4 \times 1 \times (1-c)) = 0$

$4 - 4(1-c) = 0$

$4 - 4 + 4c = 0$

$4c = 0$

$c = 0$

$\boxed{\text{TANGENT} \quad y = 2x}$

QP-11 Find the equations of the tangents from $(0, -2)$ to the curve $y = 8x^2$.

$y = 8x^2$ $\quad$ $y = mx - 2$

FOR TANGENT $b^2 - 4ac = 0$

$8x^2 = mx - 2$

$8x^2 - mx + 2 = 0$

$(-m)^2 - (4 \times 8 \times 2) = 0$

$m^2 - 64 = 0$

$m^2 = 64$

$m = \pm 8$

TANGENTS $y = 8x - 2$

$y = -8x - 2$

Polynomial Basics

A polynomial is an expression with varying powers of the same variable.

The degree of a polynomial is the value of the highest power.

$2x^4 - 3x^3 + x^2 - 5$ is a polynomial of degree 4

$5a^8 - 4a^5 + 3a^2$ is a polynomial of degree 8

A root of a polynomial, $f(x)$, is a value of x for which $f(x) = 0$

Examples

QP-12 State the degree of the following polynomials

(a) $x^5 + 6x^2 + x - 4$ (b) $3 + m - 2m^5 + m^{11}$

(a) DEGREE 5 (b) DEGREE 11

QP-13 Evaluate $f(x) = x^3 - 4x^2 + x + 6$ and state if each value is a root or not.

(a) $x = -1$ (b) $x = 0$ (c) $x = 1$ (d) $x = 2$ (e) $x = 3$

(a) $f(-1) = (-1)^3 - 4(-1)^2 + (-1) + 6$
$= 0$
So $x = -1$ IS A ROOT

(b) $f(0) = (0)^3 - 4(0)^2 + (0) + 6$
$= 6$
So $x = 0$ IS NOT A ROOT

(c) $f(1) = (1)^3 - 4(1)^2 + (1) + 6$
$= 4$
So $x = 1$ IS NOT A ROOT

(d) $f(2) = (2)^3 - 4(2)^2 + (2) + 6$
$= 0$
So $x = 2$ IS A ROOT

(e) $f(3) = (3)^3 - 4(3)^2 + (3) + 6$
$= 0$
So $x = 3$ IS A ROOT

Nested Form

A polynomial function that is arranged in descending powers can be expressed in nested form.

$$\begin{aligned} f(x) &= ax^3 + bx^2 + cx + d \\ &= (ax^2 + bx + c)x + d \\ &= ((ax + b)x + c)x + d \end{aligned}$$

This can be an easier way of evaluating a polynomial.

For example, if we use nested form to evaluate $3x^3 + 7x^2 - 2x + 3$ when $x = 4$.

$$\begin{aligned} f(x) &= 3x^3 + 7x^2 - 2x + 3 \\ &= (3x^2 + 7x - 2)x + 3 \\ &= ((3x + 7)x - 2)x + 3 \end{aligned}$$

$$\begin{aligned} f(4) &= ((3 \times 4 + 7)4 - 2)4 + 3 \\ &= (19 \times 4 - 2)4 + 3 \\ &= 74 \times 4 + 3 \\ &= 299 \end{aligned}$$

We can make this easier by laying it out like this

	x^3	x^2	x	
4	3	7	-2	3
Value to be evaluated	+	+ 12	+ 76	+ 296
	3	x4 → 19	x4 → 74	x4 → $\boxed{299}$

Examples

QP-14 Use nested form to evaluate following polynomials, for $x = 2$

(a) $2x^3 + 6x^2 - x + 2$ (b) $5x^4 - 2x^2 + 5$

(a)

	x^3	x^2	x	c
2	2	6	-1	2
		4	20	38
	2	10	19	$\boxed{40}$

(b)

	x^4	x^3	x^2	x	c
2	5	0	-2	0	5
		10	20	36	72
	5	10	18	36	$\boxed{77}$

QP-15 Use nested form to evaluate following polynomials, for $x = -1$

(a) $4x^5 - 6x^2 + 5x + 1$ (b) $2x^4 + 3x^3 - x^2 + 3x + 5$

(a)

-1	4	0	0	-6	5	1
		-4	4	-4	10	-15
	4	-4	4	-10	15	$\boxed{-14}$

(b)

-1	2	3	-1	3	5
		-2	-1	2	-5
	2	1	-2	5	$\boxed{0}$

Synthetic Division

We can divide polynomials just like we can divide numbers.

For example, $(x^2 + 4x + 3) / (x + 1) = x + 3$

This obviously increases in difficulty as the degree of the polynomial increases but we can use nested form to carry out the division. This is called synthetic division.

	x^2	x	
-1	1	4	3
Value of factor	+	+ -1	+ -3
	1	x-1 3	x-1 $\boxed{0}$

So if $f(a) = 0$ then $(x - a)$ is a factor of $f(x)$

The bottom row of the table is the quotient, what is left when the division is complete.

We must include all powers of x, even if they are blank (we use a 0)

Examples

QP-16 Use synthetic division to state the quotient and remainder for the following:

(a) $(2x^3 + 6x^2 - x + 2) / (x - 2)$

(b) $(5x^4 - 2x^2 + 5) / (x + 3)$

(a)

$x - 2 = 0$
$x = 2$

2	2	6	-1	2
		4	20	38
	2	10	19	$\boxed{40}$

QUOTIENT = $2x^2 + 10x + 19$
REMAINDER = 40

(b)

-3	5	0	-2	0	5
		-15	45	-129	387
	5	-15	43	-129	$\boxed{392}$

QUOTIENT = $5x^3 - 15x^2 + 43x - 129$
REMAINDER = 392

QP-17 Use synthetic division to fully factorise the following:

(a) $x^3 - 2x^2 - x + 2$ if $(x - 2)$ is a factor

(b) $x^3 + 3x^2 - 13x - 15$ if $(x + 1)$ is a factor

(a)

$$\begin{array}{r|rrrr} 2 & 1 & -2 & -1 & 2 \\ & & 2 & 0 & -2 \\ \hline & 1 & 0 & -1 & \boxed{0} \end{array}$$

REM = 0 SO $(x-2)$ IS A FACTOR

$$\begin{aligned} & x^3 - 2x^2 - x + 2 \\ &= (x-2)(x^2-1) \\ &= (x-2)(x+1)(x-1) \end{aligned}$$

(b)

$$\begin{array}{r|rrrr} -1 & 1 & 3 & -13 & -15 \\ & & -1 & -2 & 15 \\ \hline & 1 & 2 & -15 & \boxed{0} \end{array}$$

REM = 0 SO $(x+1)$ IS A FACTOR

$$\begin{aligned} & x^3 + 3x^2 - 13x - 15 \\ &= (x+1)(x^2+2x-15) \\ &= (x+1)(x+5)(x-3) \end{aligned}$$

Finding Factors of a Polynomial

If we are not given a factor of a polynomial, we have to find a factor by looking at the constant in the polynomial.

For example, $a^3 + 7a + 3$ then the factor could be $-1,\ 1,\ -3$ or 3.

We need to evaluate the function for each of these values until we get a value of 0.

Examples

QP-18 Fully factorise:

(a) $x^3 - x^2 - x + 1$ (b) $x^3 - 7x - 6$

(c) $x^3 - 13x - 12$ (d) $x^4 + 5x^3 + 2x^2 - 8x$

(a)

$$\begin{array}{c|cccc} 1 & 1 & -1 & -1 & 1 \\ & & 1 & 0 & -1 \\ \hline & 1 & 0 & -1 & \boxed{0} \end{array}$$

REM = 0 SO $(x-1)$ IS A FACTOR

$x^3 - x^2 - x + 1$

$= (x-1)(x^2-1)$

$= (x-1)(x-1)(x+1)$

(b)

$$\begin{array}{c|cccc} 3 & 1 & 0 & -7 & -6 \\ & & 3 & 9 & 6 \\ \hline & 1 & 3 & 2 & \boxed{0} \end{array}$$

REM = 0 SO $(x-3)$ IS A FACTOR

$x^3 - 7x - 6$

$= (x-3)(x^2+3x+2)$

$= (x-3)(x+2)(x+1)$

(c)

$$\begin{array}{c|cccc} 2 & 1 & 0 & -13 & -12 \\ & & 2 & 4 & -18 \\ \hline & 1 & 2 & -9 & \boxed{-30} \end{array}$$

REM $\neq 0$ SO $(x-2)$ IS NOT A FACTOR

$$\begin{array}{c|cccc} -1 & 1 & 0 & -13 & -12 \\ & & -1 & 1 & 12 \\ \hline & 1 & -1 & -12 & \boxed{0} \end{array}$$

REM = 0 SO $(x+1)$ IS A FACTOR

$x^3 - 13x - 12$

$= (x+1)(x^2-x-12)$

$= (x+1)(x-4)(x+3)$

(d)

$x^4 + 5x^3 + 2x^2 - 8x$

$= x(x^3 + 5x^2 + 2x - 8)$

$$\begin{array}{c|cccc} -2 & 1 & 5 & 2 & -8 \\ & & -2 & -6 & 8 \\ \hline & 1 & 3 & -4 & \boxed{0} \end{array}$$

REM = 0 SO $(x+2)$ IS A FACTOR

$x^4 + 5x^3 + 2x^2 - 8x$

$= x(x+2)(x^2+3x-4)$

$= x(x+2)(x+4)(x-1)$

Finding Unknown Coefficients of a Polynomial

We can use synthetic division to calculate an unknown coefficient in a polynomial.

Remember for factors, the remainder equals zero.

Examples

QP-19 Determine the value of the unknown coefficients:

(a) $2x^4 + 6x^3 + px^2 + 4x - 15$ if $(x + 3)$ is a factor.

(b) $2x^4 - x^3 + mx - 6$ if $(2x - 1)$ is a factor.

(c) $x^4 + ax^3 - x^2 + bx - 8$ if $(x - 2)$ and $(x + 4)$ are factors.

(a)
$$\begin{array}{r|rrrrr} -3 & 2 & 6 & p & 4 & 15 \\ & & -6 & 0 & -3p & 9p-12 \\ \hline & 2 & 0 & p & -3p+4 & \boxed{9p+3} \end{array}$$

$$9p + 3 = 0$$
$$9p = -3$$
$$p = -\frac{1}{3}$$

(b)

$(2x-1)=0$
$2x=1$
$x=\frac{1}{2}$

$$\begin{array}{r|rrrrr} \frac{1}{2} & 2 & -1 & 0 & m & -6 \\ & & 1 & 0 & 0 & \frac{1}{2}m \\ \hline & 2 & 0 & 0 & m & \boxed{\frac{1}{2}m-6} \end{array}$$

$$\frac{1}{2}m - 6 = 0$$
$$\frac{1}{2}m = 6$$
$$m = 12$$

(c)
$$\begin{array}{r|rrrrr} 2 & 1 & a & -1 & b & -8 \\ & & 2 & 2a+4 & 4a+6 & 8a+2b+12 \\ \hline & 1 & a+2 & 2a+3 & 4a+b+6 & \boxed{8a+2b+4} \end{array} = 0$$

$$\begin{array}{r|rrrrr} -4 & 1 & a & -1 & b & -8 \\ & & -4 & -4a+16 & 16a-60 & -64a-4b+240 \\ \hline & 1 & a-4 & -4a+15 & 16a+b-60 & \boxed{-64a-4b+232} \end{array} = 0$$

$$8a + 2b = -4 \quad \text{— ①} \quad \boxed{\times 2}$$
$$-64a - 4b = -232 \quad \text{— ②}$$
$$16a + 4b = -8 \quad \text{— ③}$$

②+③ $\quad -48a = -240$

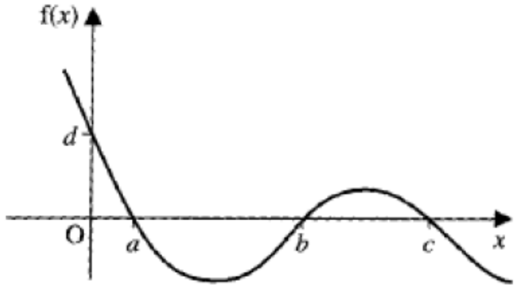
$$a = 5$$

Sub $a = 5$ into ①

$$8(5) + 2b = -4$$
$$40 + 2b = -4$$
$$2b = -44$$
$$b = -22$$

Stating the Equation of a Polynomial from its Graph

The equation of a polynomial can be found from its graph, similar to quadratics.



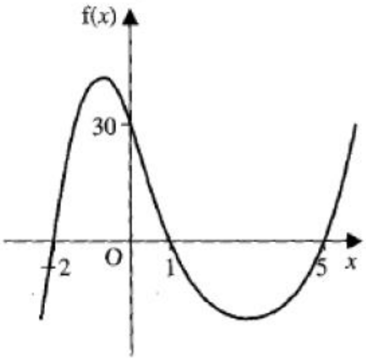
$f(x) = k(x - a)(x - b)(x - c)$ is the general equation of the function. We can calculate k by substituting in $(0, d)$.

Note that if the graph just touches the axis at $x = a$ (turning point is on the x-axis), then $(x - a)^2$ is a root.

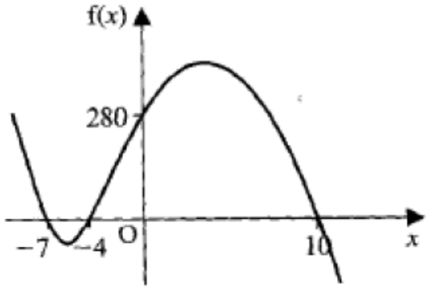
Examples

QP-20 Determine the equation of the function:

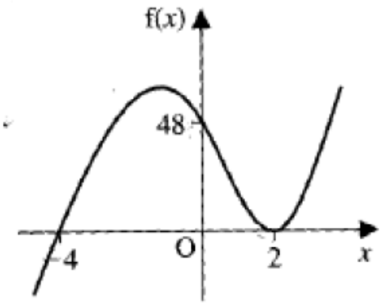
(a)



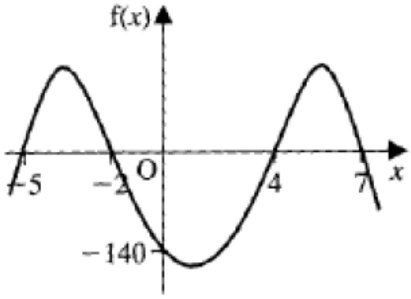
(b)



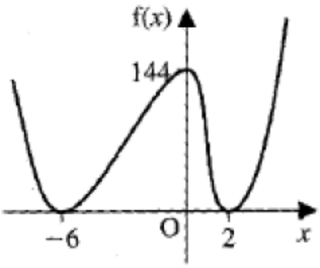
(c)



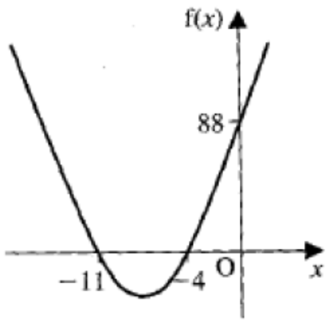
(d)



(e)



(f)



(a) $y = k(x+2)(x-1)(x-5)$

WHEN $x=0$ $y=30$

$30 = k(2)(-1)(-5)$

$30 = 10k$

$k = 3$

$y = 3(x+2)(x-1)(x-5)$

(b) $y = k(x+7)(x+4)(x-10)$

WHEN $x=0$ $y=280$

$280 = k(7)(4)(-10)$

$280 = -280k$

$k = -1$

$y = -(x+7)(x+4)(x-10)$

(c) $y = k(x+4)(x-2)(x-2)$ REPEATED ROOT

WHEN $x=0$ $y=48$

$48 = k(4)(-2)(-2)$

$48 = 16k$

$k = 3$

$y = 3(x+4)(x-2)^2$

(d) $y = k(x+5)(x+2)(x-4)(x-7)$

WHEN $x=0$ $y=-140$

$-140 = k(5)(2)(-4)(-7)$

$-140 = 280k$

$k = -\frac{1}{2}$

$y = -\frac{1}{2}(x+5)(x+2)(x-4)(x-7)$

(e) $y = k(x+6)(x+6)(x-2)(x-2)$

WHEN $x=0$ $y=144$

$144 = k(6)(6)(-2)(-2)$

$144 = 144k$

$k = 1$

$y = (x+6)^2(x-2)^2$

(f) $y = k(x+11)(x+4)$

WHEN $x=0$ $y=88$

$88 = k(11)(4)$

$88 = 44k$

$k = 2$

$y = 2(x+11)(x+4)$

Solving Polynomial Equations

Once we can factorise a polynomial, we can solve a polynomial equation.

When solving polynomial equations each bracket is equal to zero.

Examples

QP-21 Find the roots of $x^3 - 2x^2 - x + 2 = 0$

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -1 & 2 \\ & & 1 & -1 & -2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

REM = 0 SO $(x-1)$ IS A FACTOR

$$x^3 - 2x^2 - x + 2 = 0$$

$$(x-1)(x^2 - x - 2) = 0$$

$$(x-1)(x-2)(x+1) = 0$$

$$\boxed{x = 1 \quad x = 2 \quad x = -1}$$

QP-22 Solve $x^4 + 5x^3 + 2x^2 - 8x = 0$

$$x^4 + 5x^3 + 2x^2 - 8x = 0$$

$$x(x^3 + 5x^2 + 2x - 8) = 0$$

$$\begin{array}{r|rrrr} -2 & 1 & 5 & 2 & -8 \\ & & -2 & -6 & 8 \\ \hline & 1 & 3 & -4 & \boxed{0} \end{array}$$

REM = 0 SO $(x+2)$ IS A FACTOR

$$x^4 + 5x^3 + 2x^2 - 8x = 0$$

$$x(x+2)(x^2 + 3x - 4) = 0$$

$$x(x+2)(x+4)(x-1) = 0$$

$$\boxed{x = 0 \quad x = -2 \quad x = -4 \quad x = 1}$$

Summary

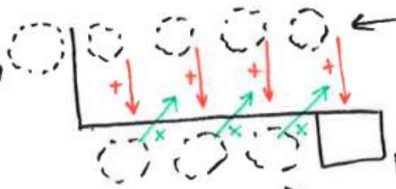
POLYNOMIAL: expression with degree ≥ 3

Polynomials and QUADRATICS

QUADRATIC: expression with degree $= 2$

FACTORISING A POLYNOMIAL:

Use synthetic division



coefficient of polynomial inc. zeros for missing terms

root

= 0 if factor

quotient (other bracket)

if $x-2$ is a factor you input 2 not -2

FINDING AN UNKNOWN IN A POLYNOMIAL:

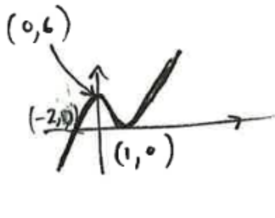
$3x^3 + 2x^2 + px + 5 = 0$. Find p if $x =$ is a factor

Exact same process as factorising but set the remainder to equal zero and solve for unknown.

GENERAL EQUATION OF A POLYNOMIAL:

$y = k(x-a)(x-b)(x-c)$ ← number brackets = number of roots

Use this to state equation of polynomial from graph.

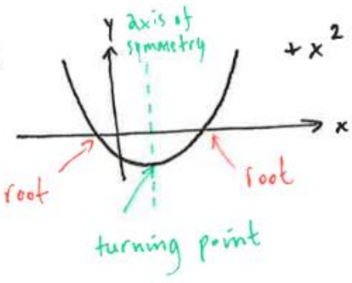


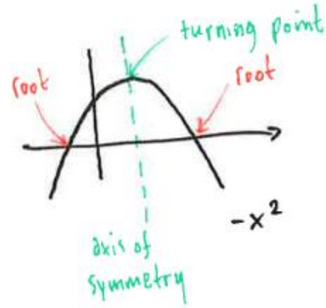
$y = k(x-a)(x-b)(x-c)$

$= k(x+2)(x-1)(x-1)$ — — — repeated root, graph only touches x-axis

Sub $y = 6$ and $x = 0$ into function to find k.

QUADRATIC GRAPHS:





axis of symmetry is halfway between roots

NATURE OF ROOTS:

use the discriminant, if $ax^2 + bx + c = 0$

$b^2 - 4ac > 0$ two real roots (cuts x-axis twice)

$b^2 - 4ac = 0$ one real root (touches x-axis once)

$b^2 - 4ac < 0$ no real roots (doesnt cut x-axis)

in addition, if $b^2 - 4ac$ is a perfect square the roots are rational

SOLVING A QUADRATIC:

If asked to solve, answer must include x =

EITHER

FACTORISE

$x^2 + bx + c = 0$

$(x + \quad)(x + \quad) = 0$

QUADRATIC FORMULA

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

memorise this

COMPLETING THE SQUARE:

Rewrite $y = ax^2 + bx + c$ in the form

$y = a(x+p)^2 + q$

AOS $x = -p$ $TP = (-p, q)$

$y = 3(x-2)^2 + 4$

TANGENT TO A PARABOLA:

(It's a touchy subject!)

1. Set parabola to equal straight line
2. Rearrange to equal zero
3. Use discriminant

It's similar to nature of roots

$b^2 - 4ac > 0$

two POI

$b^2 - 4ac = 0$

one point of contact (TANGENT)

$b^2 - 4ac < 0$

no POI