HIGHER MATHS

Integration

Notes with Examples

Mr Miscandlon
Gw13miscandlondavid@glow.sch.uk

Notation

We use $\int$ the symbol for integration.

The $\int$ must be accompanied by "dx" to show we are integrating with respect to x. This changes depending on the term we are integrating.

Rules of Integration

In integration our aim is to reverse the process of differentiation. Integration is sometime referred to as anti-derivative.

When we use the general form of integration it is referred to as an **indefinite integral.** With **indefinite integrals** we must include the constant of integration to allow for any constant that is zeroed during the differentiation process.

You can check your integration by differentiating your answer - remember it's the reverse process.

Basic Integration

The basic rule for integration is:

$$\boxed{\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c}$$

where c is the constant of integration.

Basically, increase the power by one and divide by the new power.

Examples

I-01 Find:

(a) $\int x^4\,dx$ (b) $\int x^{-2}\,dx$ (c) $\int x^{1/2}\,dx$

(a) $\int 4x\,dx$

$= \frac{4x^2}{2} + c$

$= 2x^2 + c$

(b) $\int x^{-2}\,dx$

$= \frac{x^{-1}}{-1} + c$

$= -x^{-1} + c$

$= -\frac{1}{x} + c$

(c) $\int x^{1/2}\,dx$

$= \frac{x^{3/2}}{3/2} + c$

$= \frac{2x^{3/2}}{3} + c$

$= \frac{2\sqrt{x^3}}{3} + c$

I-02 Find:

(a) $\int 2x^4\,dx$ (b) $\int 5\,dx$ (c) $\int 9x^{1/2}\,dx$

(a) $\int 2x^4\,dx$

$= \frac{2x^5}{5} + c$

(b) $\int 5\,dx$

$= 5x + c$

(c) $\int 9x^{1/2}$

$= \frac{9x^{3/2}}{3/2} + c$

$= \frac{18x^{3/2}}{3} + c$

$= 6\sqrt{x^3} + C$

Integration of Multiple Terms

The rule for integrating multiple terms is:

$$\boxed{\int (f(x) + g(x))\,dx = \int f(x) + \int g(x)}$$

Basically integrate each term separately!

Examples

I-03 Find:

(a) $\int (x^4 + 2x^3)\,dx$ (b) $\int (x^{-2} - 3x^5)\,dx$ (c) $\int (x - x^{1/2} + 5)\,dx$

(a) $\int (x^4 + 2x^3)\,dx$

$= \frac{x^5}{5} + \frac{2x^4}{4} + c$

$= \frac{x^5}{5} + \frac{x^4}{2} + c$

(b) $\int (x^{-2} - 3x^5)\,dx$

$= \frac{x^{-1}}{-1} - \frac{3x^6}{6} + c$

$= -\frac{1}{x} - \frac{x^6}{2} + c$

(c) $\int (x - x^{1/2} + 5)\,dx$

$= \frac{x^2}{2} - \frac{x^{3/2}}{3/2} + 5x + c$

$= \frac{x^2}{2} + \frac{2\sqrt{x^3}}{3} + 5x + c$

Integration of Other Variables

We can integrate any variable, the term at the end of the integral tells us what variable we are integrating with respect to.

$\int f(u)du$ is integrating with respect to u while $\int f(n)dn$ is integrating with respect to n.

Examples

I-04 Find:

(a) $\int u^4 + 2u^3\, du$ (b) $\int p^{-2} - 3p^5\, dp$ (c) $\int -q^{1/2}\, dx$

(a) $\int (u^4 + 2u^3)du$

$= \frac{u^5}{5} + \frac{2u^4}{4} + c$

$= \frac{u^5}{5} + \frac{u^4}{2} + c$

(b) $\int (p^{-2} - 3p^5)\, dp$

$= \frac{p^{-1}}{-1} - \frac{3p^6}{6} + c$

$= -\frac{1}{p} - \frac{p^6}{2} + c$

(c) $\int -q^{1/2}\, dx$

$= -q^{1/2}x + c$

THINK

$\int 5\, dx$

$= 5x + c$

Preparing to Integrate

It is often necessary to use the index rules to prepare a function for integration. The integration rules will only work with powers of x. In order to integrate, you must be able to convert a function involving roots, fractions, brackets etc. into a sum or difference of powers of x.

If necessary, use the index rules to give the integral using positive indices. There are normally no marks awarded for changing the form of an integral once you have obtained it. However, you often have to go further and evaluate the integral and will find this easier using positive indices.

Examples

I-05 Find:

(a) $\int \frac{dx}{x^4}$ (b) $\int \frac{dx}{\sqrt{x}}$ (c) $\int \frac{5}{2\sqrt{x}}\, dx$ (d) $\int \frac{4p^3 - p}{3}\, dp$

(a) $\int \frac{dx}{x^4}$

$= \int x^{-4}\, dx$

$= \frac{x^{-3}}{-3} + c$

$= -\frac{x^{-3}}{3} + c$

(b) $\int \frac{dx}{\sqrt{x}}$

$= \int x^{-1/2}\, dx$

$= \frac{x^{1/2}}{1/2} + c$

$= 2\sqrt{x} + c$

(c) $\int \frac{5}{2\sqrt{x}}\, dx$

$= \int \frac{5x^{-1/2}}{2}\, dx$

$= \frac{5x^{1/2}}{2 \times 1/2} + c$

$= 5\sqrt{x} + c$

(d) $\int \frac{4p^3 - p}{3}\, dp$

$= \int \frac{4p^3}{3} - \frac{p}{3}\, dp$

$= \frac{4p^4}{3 \times 4} - \frac{p^2}{3 \times 2} + c$

$= \frac{p^4}{3} - \frac{p^2}{6} + c$

Differential Equations

A differential equation is one that contains derivatives. To solve a differential equation we use integration to get the general solution then use the additional information (usually an x and y value) to find the specific solution.

Examples

I-06 Find the particular solution of the differential equation $\frac{dy}{dx} = 8x - 1$ given that $y = 5$ when $x = 1$.

$$\frac{dy}{dx} = 8x - 1$$
$$y = \frac{8x^2}{2} - x + c$$
$$= 4x^2 - x + c$$

WHEN $x = 1$ $y = 5$

$$5 = 4(1)^2 - (1) + c$$
$$5 = 4 - 1 + c$$
$$5 = 3 + c$$
$$c = 2$$

$$y = 4x^2 - x + 2$$

I-07 The gradient of a tangent to a curve is given by $\frac{dy}{dx} = 2x - 3$. If the curve passes through the point (4, 3), find its equation.

$$\frac{dy}{dx} = 2x - 3$$
$$y = \frac{2x^2}{2} - 3x + c$$
$$= x^2 - 3x + c$$

WHEN $x = 4$ $y = 3$

$$3 = (4)^2 - 3(4) + c$$
$$3 = 16 - 12 + c$$
$$3 = 4 + c$$
$$c = -1$$

$$y = x^2 - 3x - 1$$

I-08 The function f, defined on a suitable domain, is such that $f'(x) = x^2 + \frac{1}{x^2} + \frac{2}{3}$

Given that $f(1) = 4$, find a formula for $f(x)$.

$$f'(x) = x^2 + \frac{1}{x^2} + \frac{2}{3}$$
$$f'(x) = x^2 + x^{-2} + \frac{2}{3}$$
$$f(x) = \frac{x^3}{3} + \frac{x^{-1}}{-1} + \frac{2}{3}x + c$$
$$= \frac{x^3}{3} - \frac{1}{x} + \frac{2}{3}x + c$$

WHEN $x = 1$ $f(x) = 4$

$$4 = \frac{(1)^3}{3} - \frac{1}{1} + \frac{2}{3}(1) + c$$
$$4 = 0 + c$$
$$c = -4$$

$$f(x) = \frac{x^3}{3} - \frac{1}{x} + \frac{2}{3}x - 4$$

Definite Integrals

A definite integral is when we evaluate an integral between two limits. A definite integral is a **numerical value.**

If $F(x)$ is the integral of $f(x)$ then

$$\boxed{\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)}$$

where b is the **upper limit** and a is the **lower limit.**

To calculate a definite interval we integrate (without the constant of integration), evaluate the integral at the upper limit, evaluate the integral at the lower limit then subtract the lower limit value from the upper limit value.

Examples

I-09 Find:

(a) $\int_1^3 x^4\,dx$ (b) $\int_0^4 5x^2\,dx$ (c) $\int_{-1}^2 (x^4 + 2x^{-3})\,dx$

(a) $\int_1^3 x^4\,dx$

$= \left[\frac{x^5}{5}\right]_1^3$

$= \left(\frac{(3)^5}{5}\right) - \left(\frac{(1)^5}{5}\right)$

$= \frac{243}{5} - \frac{1}{5}$

$= \frac{242}{5}$

(b) $\int_1^4 5x^{-2}\,dx$

$= \left[\frac{5x^{-1}}{-1}\right]_1^4$

$= \left[-\frac{5}{x}\right]_1^4$

$= \left(-\frac{5}{(4)}\right) - \left(-\frac{5}{1}\right)$

$= \frac{15}{4}$

(c) $\int_{-1}^2 (x^4 + 2x^3)\,dx$

$= \left[\frac{x^5}{5} + \frac{2x^4}{4}\right]_{-1}^2$

$= \left[\frac{x^5}{5} + \frac{x^4}{2}\right]_{-1}^2$

$= \left(\frac{(2)^5}{5} + \frac{(2)^4}{2}\right) - \left(\frac{(-1)^5}{5} + \frac{(-1)^4}{2}\right)$

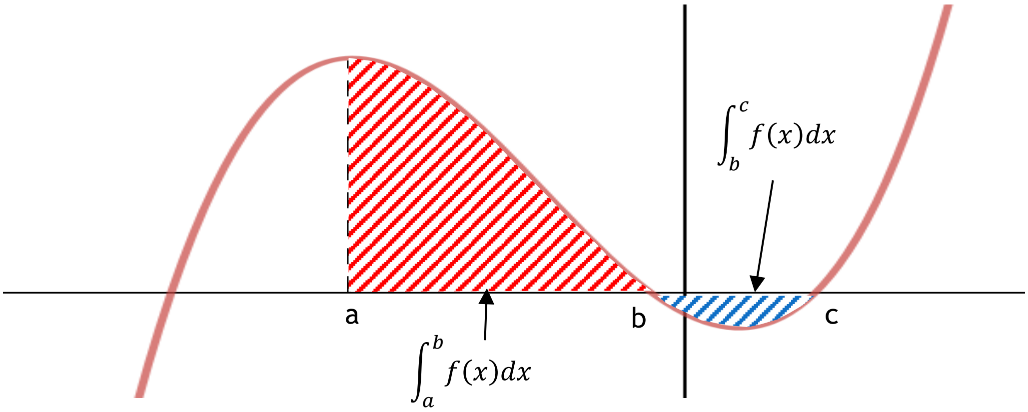
$= \frac{141}{10}$

Area between a Curve and the x-axis

Integration can be used to calculate the area between a curve and the x-axis.

The area between the graph of $y = f(x)$ and the x-axis between $x = a$ and $x = b$ is given by:

$$\boxed{Area = \int_a^b f(x)dx}$$



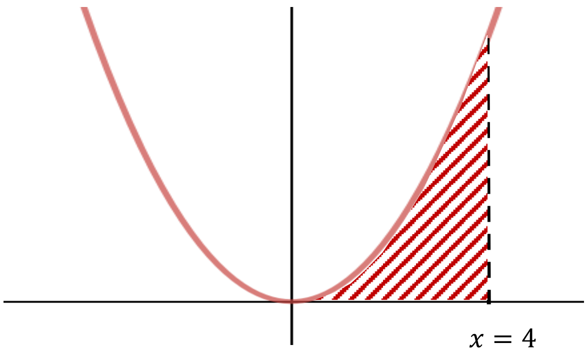
If the area is **above** the x-axis the definite integral will be **positive**.

If the area is **below** the x-axis the definite integral will be **negative**. In this case we **ignore** the negative sign.

If the area is above **and** below the x-axis we **MUST** integrate the parts above and below the x-axis **separately**, ignore the negative and add the values.

Examples

I-10 Find the area under the curve $y = 2x^2$ from $x = 0$ to $x = 4$.



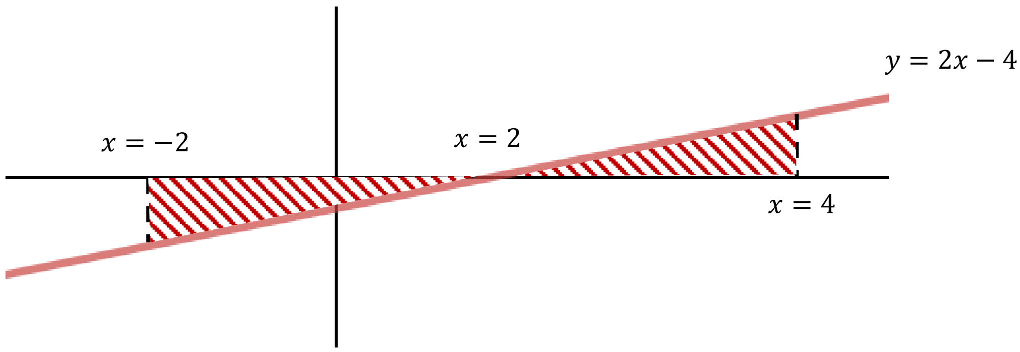
$$\text{AREA} = \int_0^4 2x^2\,dx$$

$$= \left[\frac{2x^3}{3}\right]_0^4$$

$$= \left(\frac{2(4)^3}{3}\right) - (0)$$

$$= \frac{128}{3} \text{ units}^2$$

I-11 Calculate the total shaded area in the diagram.



AREA BELOW AXIS $= \int_{-2}^{2} 2x - 4 \, dx$

$= \left[\frac{2x^2}{2} - 4x \right]_{-2}^{2}$

$= \left[x^2 - 4x \right]_{-2}^{2}$

$= \left((2^2) - 4(2)\right) - \left((-2)^2 - 4(-2)\right)$

$= -16$

$= 16 \text{units}^2$

AREA ABOVE AXIS $= \int_{2}^{4} 2x - 4 \, dx$

$= \left[x^2 - 4x \right]_{2}^{4}$

$= \left((4)^2 - 4(4)\right) - \left((2)^2 - 4(2)\right)$

$= 4 \text{ units}^2$

$$\boxed{\begin{aligned} \text{TOTAL AREA} &= 16 + 4 \\ &= 20 \text{ UNITS}^2 \end{aligned}}$$

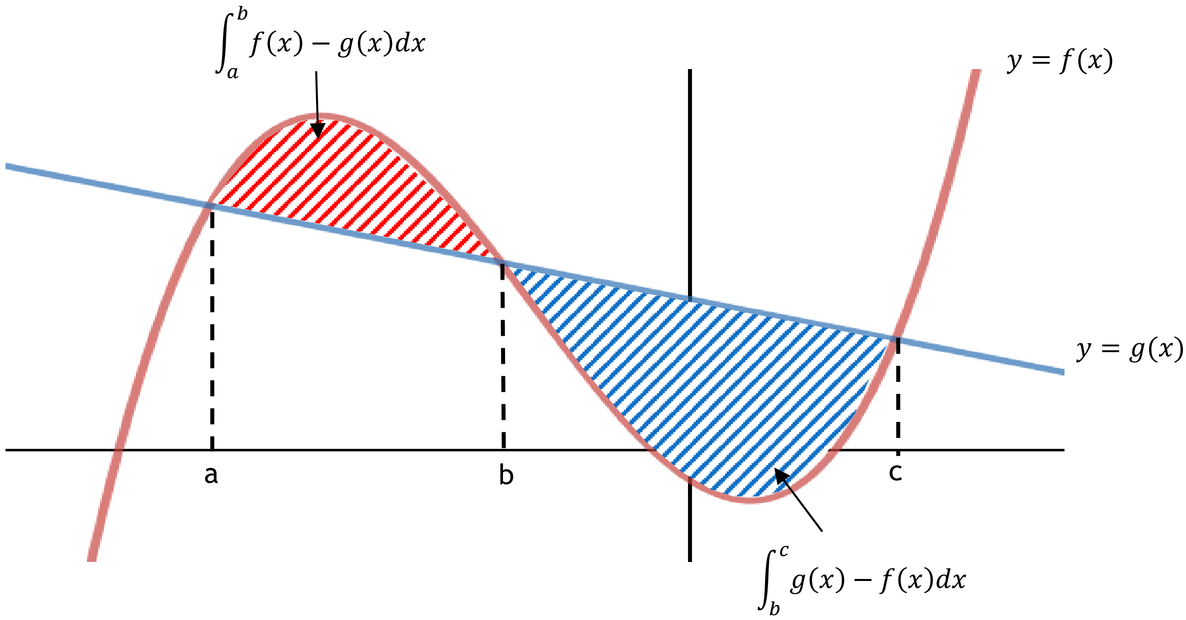
Area between two Curves

Integration can be used to calculate the area between two curves (or a curve and a straight line).

The area between the graph of $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$ is given by:

$$\boxed{Area = \int_a^b upper\ function - lower\ function}$$

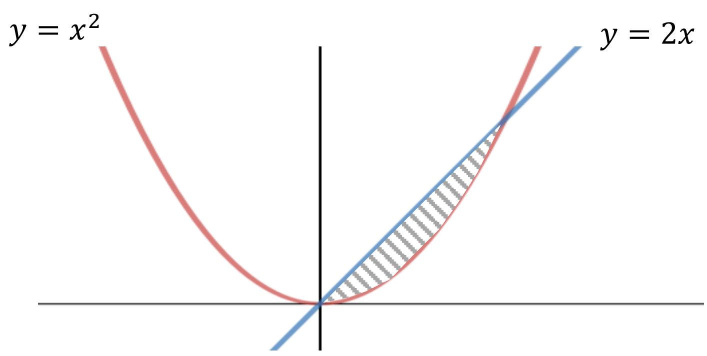
where the upper function is the top curve and the lower function is the bottom curve.



Note that we do not need to worry about if the area is above or below that x-axis when dealing with area between two curves.

Examples

I-12 Calculate the area enclosed by the parabola $y = x^2$ and the line $y = 2x$.



PoI :

$x^2 = 2x$

$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0 \quad x = 2$

$$\text{AREA} = \int \text{UPPER} - \text{LOWER}$$
$$= \int_0^2 2x - x^2 \, dx$$
$$= \left[x^2 - \frac{x^3}{3} \right]_0^2$$
$$= \left((2)^2 - \frac{(2)^3}{3} \right) - (0)$$
$$= \frac{4}{3} \text{ units}^2$$

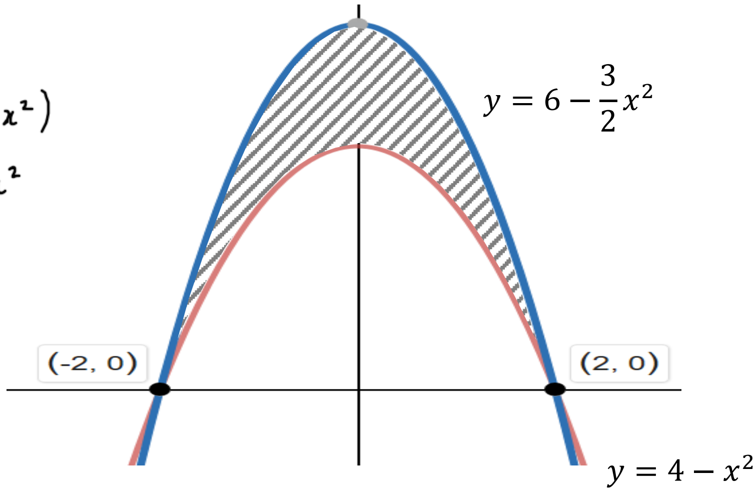
I-13 Calculate the area enclosed by the graphs of $y = 4 - x^2$ and $y = 6 - \frac{3}{2}x^2$.

UPPER − LOWER

$= 6 - \frac{3}{2}x^2 - (4 - x^2)$

$= 6 - \frac{3}{2}x^2 - 4 + x^2$

$= 2 - \frac{x^2}{2}$



$\text{AREA} = \int_{-2}^{2} 2 - \frac{x^2}{2}\,dx$

$= \left[2x - \frac{x^3}{6}\right]_{-2}^{2}$

$= \left(2(2) - \frac{(2)^3}{6}\right) - \left(2(-2) - \frac{(-2)^3}{6}\right)$

$= \frac{32}{6}$

$= \frac{16}{3}$ units^2

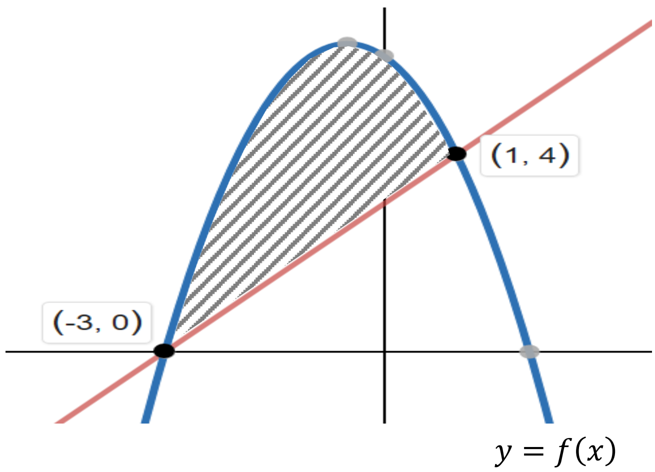
I-14 Calculate the area enclosed by the functions $g(x) = x + 3$ and $f(x) = 6 - x - x^2$.

UPPER − LOWER

$= (6 - x - x^2) - (x + 3)$

$= 6 - x - x^2 - x - 3$

$= 3 - 2x - x^2$



$\text{AREA} = \int_{-3}^{1} 3 - 2x - x^2\,dx$

$= \left[3x - x^2 - \frac{x^3}{3}\right]_{-3}^{1}$

$= \left(3(1) - (1)^2 - \frac{(1)^3}{3}\right) - \left(3(-3) - (-3)^2 - \frac{(-3)^3}{3}\right)$

$= \frac{32}{3}$ units^2

I-15 Calculate the area enclosed in the diagram.

POI:

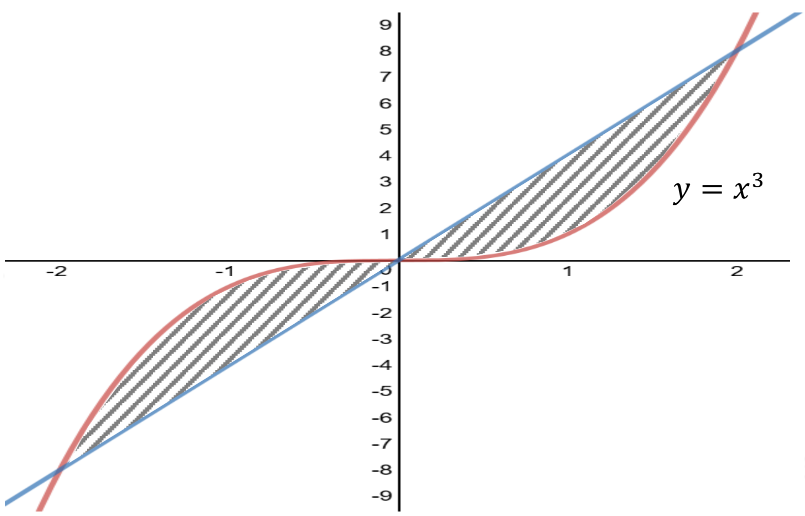
$x^3 = 4x$

$x^3 - 4x = 0$

$x(x^2 - 4) = 0$

$x = 0 \quad x^2 = 4$

$x = \pm 2$



$\int$ UPPER − LOWER

$= \int_0^2 4x - x^3\,dx$

$= \left[2x^2 - \frac{x^4}{4}\right]_0^2$

$= \left(2(2)^2 - \frac{(2)^4}{4}\right) - (0)$

$= 4$

$\boxed{\text{TOTAL AREA} = 4 \times 2 = 8 \text{ units}^2}$

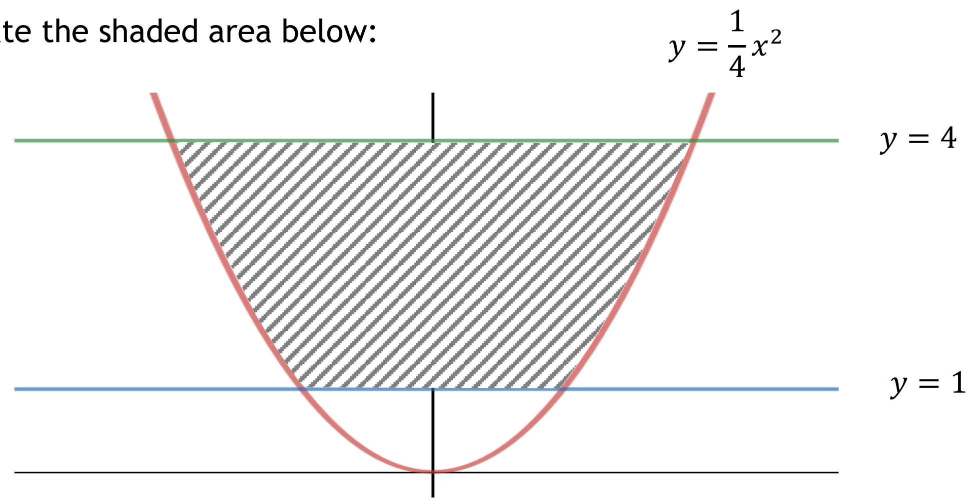
Integrating along the y-axis

It is sometimes easier to find a shaded area by integrating with respect to y instead of x.

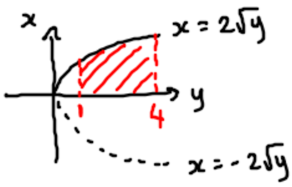
If this is the case, rearrange the equation to the form $x =$, then integrate.

Examples

I-16 Calculate the shaded area below:



$$\frac{1}{4}x^2 = y$$
$$x^2 = 4y$$
$$x = \pm 2\sqrt{y}$$



$$\int_1^4 2\sqrt{y}\ dy$$
$$= \int_1^4 2y^{1/2}\ dy$$
$$= \left[\frac{4y^{3/2}}{3}\right]_1^4$$
$$= \left[\frac{4\sqrt{y^3}}{3}\right]_1^4$$
$$= \left(\frac{4\sqrt{(4)^3}}{3}\right) - \left(\frac{4\sqrt{(1)^3}}{3}\right)$$
$$= \frac{28}{3}$$

$$\boxed{\begin{array}{l}\text{TOTAL AREA} \\ \frac{28}{3} \times 2 \\ = \frac{56}{3} \text{ units}^2\end{array}}$$

Summary

INTEGRATION
(OPPOSITE OF DIFFERENTIATION)

RULES:

$$\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + C$$

dont forget the constant of integration

$$\int 5x^2 + 2x + 3 \, dx$$
$$= \frac{5x^3}{3} + \frac{2x^2}{2} + 3x + C$$
$$= \frac{5x^3}{3} + x^2 + 3x + C$$

tidy up!

" increase the power then divide by the new power "

PREPARING TO INTEGRATE:

We can't integrate roots, brackets or x as a denominator

$$\sqrt[3]{x^2} \Rightarrow x^{2/3}$$
$$x(x+2) \Rightarrow x^2 + 2x$$
$$\frac{3}{x^2} \Rightarrow 3x^{-2}$$

THIS MUST BE DONE BEFORE INTEGRATING.

INTEGRATING WITH LIMITS:

Answer will be a number not an expression!

$$\int_1^3 x \, dx$$

limits: 3 and 1

$$= \left[\frac{x^2}{2}\right]_1^3$$

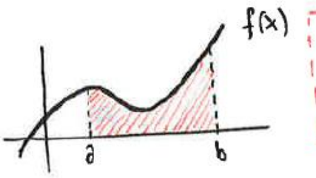
notice "c" doesn't appear

$$= \left(\frac{(3)^2}{2}\right) - \left(\frac{(1)^2}{2}\right)$$

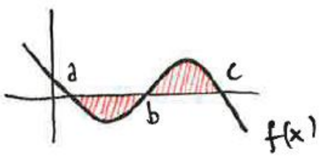
evaluate when x=3 | evaluate when x=1

$$= \frac{9}{2} - \frac{1}{2}$$
$$= 4$$

AREA UNDER A CURVE:



$$\int_a^b f(x) \, dx$$



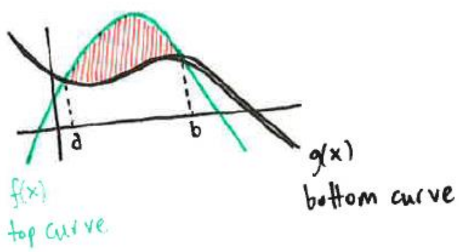
$$\int_a^b f(x) \, dx$$ (ignore neg value)

$$+$$

$$\int_b^c f(x) \, dx$$

REMEMBER UNITS : UNITS^2

AREA BETWEEN TWO CURVES:
(OR A CURVE & STRAIGHT LINE)



$$\int_a^b f(x) - g(x) \, dx$$

$$\int_a^b \text{upper} - \text{lower} \, dx$$

KEY WORDS:

" integrate " " area under | between "
" calculate f(x) if f'(x) = " " calculate y if $\frac{dy}{dx}$ = "