HIGHER MATHS

Further Calculus

Notes with Examples

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Differentiating sin x and cos x

In order to differentiate expressions that include trigonometric functions we must follow these two rules (these are given in an assessment).

$$f(x) = \sin ax$$
$$f'(x) = a\cos ax$$

$$g(x) = \cos ax$$
$$g'(x) = -a\sin ax$$

When we differentiate trigonometric functions we **must** use radians.

Examples

FC-01 Differentiate $y = 4\cos x$ with respect to x.

$$y = 4\cos x$$
$$\frac{dy}{dx} = -4\sin x$$

FC-02 Find the equation of the tangent to the curve $y = \cos x$ when $x = \frac{\pi}{3}$.

$$y = \cos x$$
$$\frac{dy}{dx} = -\sin x$$

FOR POINT :

$$y = \cos\frac{\pi}{3}$$
$$= \frac{1}{2}$$
$$\boxed{\text{POC } \left(\frac{\pi}{3}, \frac{1}{2}\right)}$$

FOR GRADIENT :

$$\frac{dy}{dx} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$
$$\boxed{m = \frac{-\sqrt{3}}{2}}$$

EQUATION :

$$y - \frac{1}{2} = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)$$
$$2y - 1 = -\sqrt{3}x + \frac{\sqrt{3}\pi}{3}$$
$$\sqrt{3}x + 2y - 1 - \frac{\sqrt{3}\pi}{3} = 0$$

FC-03 A function is defined by $f(x) = 2\sin x + 4\cos x$. Find $f'(\frac{\pi}{6})$.

$f(x) = 2\sin x + 4\cos x$

$f'(x) = 2\cos x - 4\sin x$

$f'\left(\frac{\pi}{6}\right) = 2\cos\frac{\pi}{6} - 4\sin\frac{\pi}{6}$

$= 2 \times \frac{\sqrt{3}}{2} - 4 \times \frac{1}{2}$

$= \frac{2\sqrt{3}}{2} - 2$

$= \sqrt{3} - 2$

FC-04 Differentiate $y = \sin(4x - \frac{\pi}{2})$.

$y = \sin\left(4x - \frac{\pi}{2}\right)$

$\frac{dy}{dx} = 4\cos\left(4x - \frac{\pi}{2}\right)$

The Chain Rule

It is possible to differentiate composite functions such as $f(g(x))$ using the chain rule. The chain rule is:

$$\boxed{\frac{d}{dx}[f(g(x))] = f'(g(x)) \times g'(x)}$$

Basically differentiate the outer function, leave the bracket and multiply by the derivative of the bracket.

This rule is **not given** in an assessment.

Examples

FC-05 Differentiate $y = (3u - 5)^4$ with respect to u.

$$\begin{aligned} y &= (3u-5)^4 \\ &= 4(3u-5)^3 \times 3 \\ &= 12(3u-5)^3 \end{aligned}$$

FC-06 A function is defined as $f(x) = 3sin^5x$. Find $f'(x)$.

$$\begin{aligned} f(x) &= 3\sin^5 x \\ &= 3(\sin x)^5 \\ f'(x) &= 15(\sin x)^4 \times \cos x \\ &= 15\cos x \sin^4 x \end{aligned}$$

FC-07 Differentiate $y = \sqrt{5x^2 + 7}$ with respect to x.

$$\begin{aligned} y &= \sqrt{5x^2+7} \\ &= (5x^2+7)^{1/2} \\ \frac{dy}{dx} &= \frac{1}{2}(5x^2+7)^{-1/2} \times 10x \\ &= 5x(5x^2+7)^{-1/2} \\ &= \frac{5x}{\sqrt{(5x^2+7)}} \end{aligned}$$

FC-08 A function is defined as $f(x) = \frac{1}{(3x-2)^4}$. Find $f'(1)$.

$$f(x) = \frac{1}{(3x-2)^4}$$
$$= (3x-2)^{-4}$$
$$f'(x) = -4(3x-2)^{-5} \times 3$$
$$= -12(3x-2)^{-5}$$
$$= \frac{-12}{(3x-2)^5}$$

$$f'(1) = \frac{-12}{(3(1)-2)^5}$$
$$= \frac{-12}{(1)^5}$$
$$= -12$$

FC-09 Differentiate $y = \sqrt[3]{(4x+3)^5}$ with respect to x.

$$y = \sqrt[3]{(4x+3)^5}$$
$$= (4x+3)^{5/3}$$
$$\frac{dy}{dx} = \frac{5}{3}(4x+3)^{2/3} \times 4$$
$$= \frac{20}{3}(4x+3)^{2/3}$$
$$= \frac{20}{3}\sqrt[3]{(4x+3)^2}$$

Integrating sin x and cos x

In order to integrate expressions that include trigonometric functions we must follow these two rules (these are given in an assessment).

$$\int \sin ax = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax = \frac{1}{a}\sin ax + C$$

When we integrate trigonometric functions we **must** use radians.

Examples

FC-10 Find $\int (3\sin x + 5\cos x)dx$.

$$\int (3\sin x + 5\cos x)\,dx$$

$$= -3\cos x + 5\sin x + C$$

FC-11 Find $\int_0^{\frac{\pi}{3}} (3\cos x - 2\sin x)dx$.

$$\int_0^{\pi/3} (3\cos x - 2\sin x)\,dx$$

$$= \left[3\sin x + 2\cos x\right]_0^{\pi/3}$$

$$= \left(3\sin\frac{\pi}{3} + 2\cos\frac{\pi}{3}\right) - \left(3\sin 0 + 2\cos 0\right)$$

$$= \left(\frac{3\sqrt{3}}{2} + 2\left(\frac{1}{2}\right)\right) - \left(3(0) + 2(1)\right)$$

$$= \frac{3\sqrt{3}}{2} + 1 - 2$$

$$= \frac{3\sqrt{3}}{2} - 1$$

FC-12 Find the value of $\int_0^2 \left(\frac{1}{4}\sin x\right) dx$.

$$\int_0^2 \left(\frac{1}{4}\sin x\right) dx$$

$$= \left[-\frac{1}{4}\cos x\right]_0^2$$

$$= \left(-\frac{1}{4}\cos 2\right) - \left(-\frac{1}{4}\cos 0\right)$$

$$= \left(-\frac{1}{4} \times -0.416\right) - \left(-0.25\right)$$

$$= 0.354$$

remember this is radians

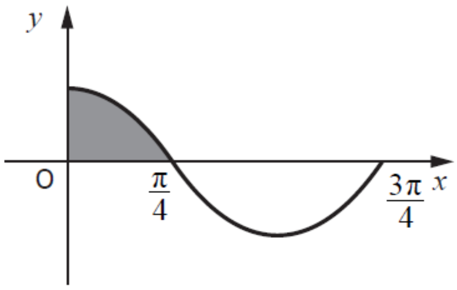
EITHER change calculator mode or convert to degrees

$2 \text{ rads} = 2 \times 180 \div \pi$

$= 114.6°$

FC-13 The diagram shows part of the graph of $y = a\cos bx$.

The shaded area is $\frac{1}{2}$ unit^2.



What is the value of $\int_0^{\frac{3\pi}{4}} (a\cos bx) dx$?

AREA ABOVE AXIS $= \frac{1}{2}$

AREA BELOW AXIS $= -1$

$$\boxed{\int_0^{\frac{3\pi}{4}} (a\cos bx)\, dx = -\frac{1}{2}}$$

Special Integrals

It is possible to integrate functions in the form of $(ax+b)^n$ by following this rule.

$$\boxed{\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C}$$

Basically increase the power by one and divide by the new power, leave the bracket and divide by the derivative of the bracket.

This rule is **not given** in an assessment.

Examples

FC-14 Find $\int (x-5)^6\, dx$.

$$\int (x-5)^6\, dx$$
$$= \frac{(x-1)^7}{7} \times \frac{1}{1}$$
$$= \frac{(x-5)^7}{7} + C$$

FC-15 Find $\int (4x+7)^5\, dx$.

$$\int (4x+7)^5\, dx$$
$$= \frac{(4x+7)^6}{6} \times \frac{1}{4}$$
$$= \frac{(4x+7)^6}{24} + C$$

FC-16 Find $\int \frac{1}{\sqrt{(4x+7)}}\, dx$.

$$\int \frac{1}{\sqrt{4x+7}}\, dx$$
$$= \int (4x+7)^{-1/2}\, dx$$
$$= \frac{(4x+7)^{1/2}}{1/2} \times \frac{1}{4} + C$$
$$= \frac{\sqrt{4x+7}}{2} + C$$

FC-17 Evaluate $\int_0^3 \frac{1}{\sqrt{(3x+7)^3}}\,dx$.

$$\int_0^3 \frac{1}{\sqrt{(3x+7)^3}}\,dx$$

$$= \int_0^3 (3x+7)^{-3/2}\,dx$$

$$= \left[\frac{(3x+7)^{-1/2}}{-1/2} \times \frac{1}{3} \right]_0^3$$

$$= \left[-\frac{2}{3\sqrt{3x+7}} \right]_0^3$$

$$= \left(\frac{-2}{3\sqrt{3(3)+7}} \right) - \left(\frac{-2}{3\sqrt{3(0)+7}} \right)$$

$$= -\frac{2}{12} + \frac{2}{3\sqrt{7}}$$

$$= \frac{2}{3\sqrt{7}} - \frac{1}{6}$$

Using Differentiation to Integrate

Because integration is the reverse process of differentiation then we can use differentiation to integrate complex functions.

Examples

FC-18 (a) Given that $y = (x^2+7)^{\frac{1}{2}}$, find $\frac{dy}{dx}$.

(b) Hence find $\int \frac{4x}{\sqrt{x^2+7}}\,dx$.

(a) $y = (x^2+7)^{1/2}$

$\frac{dy}{dx} = \frac{1}{2}(x^2+7)^{-1/2} \times 2x$

$= \frac{x}{\sqrt{x^2+7}}$

(b) $\int \frac{4}{\sqrt{x^2+7}}\,dx$

$= \int 4(x^2+7)^{-1/2}\,dx$

$= 4(x^2+7)^{1/2} + c$

FURTHER CALCULUS

DIFFERENTIATION

TRIG:

$y = \sin x$

$\frac{dy}{dx} = \cos x$

$y = \cos x$

$\frac{dy}{dx} = -\sin x$

GIVEN IN FORMULAE SHEET

REMEMBER: ALL ANGLES MUST BE IN RADIANS!!!

CHAIN RULE:

$$y = (ax+b)^n$$

$$\frac{dy}{dx} = n(ax+b)^{n-1} \times a$$

"multiply by power then decrease power by 1. also multiply by derivative of the bracket"

$$y = (4x+3)^5$$

$$\frac{dy}{dx} = 5(4x+3)^4 \times 4$$

$$= 20(4x+3)^4$$

$$y = (x^3+4)^7$$

$$\frac{dy}{dx} = 7(x^3+4)^6 \times 3x^2$$

$$= 21x^2(x^3+4)^6$$

INTEGRATION

TRIG:

$\int \sin x \, dx$

$= -\cos x + c$

$\int \cos x \, dx$

$= \sin x + c$

$\int \sin(ax+b) \, dx$

$= -\frac{1}{a}\cos(ax+b) + c$

$\int \cos(ax+b) \, dx$

$= \frac{1}{a}\sin(ax+b) + c$

INTEGRATE $(ax+b)^n$:

$$\int (ax+b)^n \, dx$$

$$= \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

"increase the power by 1 and divide by new power and derivative of bracket"

$$\int (4x+2)^5 \, dx$$

$$= \frac{(4x+2)^6}{6 \times 4} + c$$

$$= \frac{(4x+2)^6}{24} + c$$

$$\int (3x^2+1)^7 \, dx$$

$$= \frac{(3x^2+1)^8}{8 \times 6x} + c$$

$$= \frac{(3x^2+1)^8}{48x} + c$$