HIGHER MATHS

Functions & Graphs

Notes with Examples

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Set notation

In Maths we use shorthand symbols to describe sets of numbers or objects.

For Example:

$A = \{x \in R: 1 \leq x \leq 5\}$ is the shorthand for 'A is the set of Real numbers from 1 to 5 inclusive'.

Notation	Example
$\in$ means 'is a member of'	$4 \in \{1, 2, 3, 4\}$
$\notin$ means 'is not a member of'	$5 \notin \{1, 2, 3, 4\}$

Some standard sets are:

The set of **natural numbers, N** = {1,2,3,4,5,…}

The set of **whole numbers**, **W** = {0,1,2,3,4,5,…}

The set of **integers, Z** = {…,-3,-2,-1,0,1,2,3,…}

The set of **rational numbers, Q**, is the set of all numbers that can be written as fractions

The set of **real numbers, R**, is the set of numbers - rational and irrational

Functions

Function: a special relationship between two sets, A and B, where each member of set A (inputs) has a unique image in set B (outputs).

Domain: set of inputs

Range: set of outputs

Most functions allow for any value of x to be input. However, there are two special functions that the domains have to be restricted - algebraic fractions (denominator can't be zero) and square roots (bit under root can't be less than zero).

Examples

F-01 The function f is defined by $f(x) = \cos x$. State the range of the function.

RANGE $\{x \in R : -1 < x < 1\}$

F-02 State a suitable domain for the following functions

(i) $h(x) = \sqrt{x}$ (ii) $k(x) = \frac{1}{x-1}$ (iii) $p(x) = \frac{1}{x(x+1)}$

(i) $\{x \in R : x \geq 0\}$

(ii) $x - 1 \neq 0$

$x \neq 1$

$\{x \in R : x \neq 1\}$

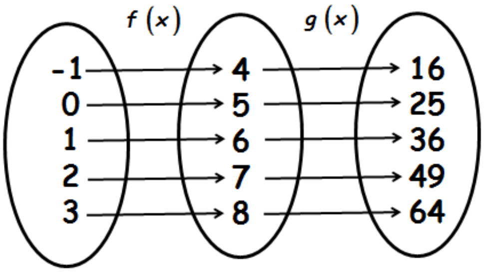
(iii) $x + 1 \neq 0$

$x \neq -1$

$\{x \in R : x \neq -1 \,;\, x \neq 0\}$

Composite function

Assume we have a two-stage calculation as shown below.



$f(x) = x + 5$

$g(x) = x^2$

We first take the domain and substitute them into $f(x)$. We then take the answers and put them into $g(x)$. This is called composition of functions.

If we were to find a single function $h(x)$ that could carry out the calculation in one step, $h(x)$ would be a composite function.

$h(x) = (x+5)^2$ $h(x) = g(f(x))$ "g of f of x"

Note $f(g(x)) \neq g(f(x))$

Examples

F-03 $f(x) = x^3$, $g(x) = 3x$. Calculate $f(g(1))$ and $g(f(-2))$.

$$f(g(x)) = f(3x) = (3x)^3$$
$$f(g(1)) = (3(1))^3 = 3^3 = 27$$

$$g(f(x)) = g(x^3) = 3x^3$$
$$g(f(-2)) = 3(-2)^3 = -24$$

F-04 If $f(x) = x - 2$ and $g(x) = x^2$, then find:

(i) $f(g(x))$ (ii) $g(f(x))$ (iii) $f(f(x))$ (iv) $g(g(x))$

$$f(g(x)) = f(x^2) = x^2 - 2$$

$$g(f(x)) = g(x-2) = (x-2)^2$$

$$f(f(x)) = f(x-2) = (x-2) - 2 = x - 4$$

$$g(g(x)) = g(x^2) = (x^2)^2 = x^4$$

F-05 If $(x) = \frac{x}{x-1}$, find a formula for $f(f(x))$.

$$f(f(x)) = f\left(\frac{x}{x-1}\right) = \frac{\left(\frac{x}{x-1}\right)}{\left(\frac{x}{x-1}\right) - 1} = \frac{\left(\frac{x}{x-1}\right)}{\frac{x}{x-1} - \frac{x-1}{x-1}} = \frac{\left(\frac{x}{x-1}\right)}{\left(\frac{1}{x-1}\right)} = \frac{x}{(x-1)} \times \frac{(x-1)}{1} = x$$

F-06 If $f(x) = cos\, x$ and $g(x) = 1 - x^2$, then find:

(i) $f(g(x))$ (ii) $g(f(x))$

$f(g(x)) = f(1 - x^2)$
$= \cos(1 - x^2)$

$g(f(x)) = g(\cos x)$
$= 1 - (\cos x)^2$
$= 1 - \cos^2 x$

F-07 If $f(x) = 2x$, $g(x) = x^2$ and $h(x) = 4x - 2$ find

(i) $f\left(g(h(x))\right)$ (ii) $h(g(f(x)))$

$g(h(x)) = g(4x - 2)$
$= (4x - 2)^2$

$f(g(h(x))) = f((4x - 2)^2)$
$= 2(4x - 2)^2$

$g(f(x)) = g(2x)$
$= (2x)^2$
$= 4x^2$

$h(g(f(x))) = h(4x^2)$
$= 4(4x^2) - 2$
$= 16x^2 - 2$

Inverse function

A function where each value in the domain has a unique value in the range is said to have one-to-one correspondence. When a function f is a one-to-one correspondence from set A to set B then another function f^{-1} exists mapping set B to set A. This is called the inverse function.

It should be noted that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

Examples

F-08 Calculate the inverse functions of

(i) $f(x) = 2x$ (ii) $f(x) = 2x^3 + 1$

(i)

$y = 2x$

$2x = y$

$x = \frac{y}{2}$

$f^{-1}(x) = \frac{x}{2}$

(ii)

$y = 2x^3 + 1$

$2x^3 + 1 = y$

$2x^3 = y - 1$

$x^3 = \frac{y-1}{2}$

$x = \sqrt[3]{\frac{y-1}{2}}$

$f^{-1}(x) = \sqrt[3]{\frac{x-1}{2}}$

F-09 If $f(x) = \frac{x-3}{4}$ find

(i) $f^{-1}(x)$ (ii) $f(f^{-1}(x))$ (iii) $f^{-1}(f(x))$

(i) $y = \frac{x-3}{4}$

$\frac{x-3}{4} = y$

$x - 3 = 4y$

$x = 4y + 3$

$f^{-1}(x) = 4x + 3$

(ii) $f(f^{-1}(x)) = f(4x + 3)$

$= \frac{(4x+3)-3}{4}$

$= \frac{4x}{4}$

$= x$

(iii) $f^{-1}(f(x)) = f\left(\frac{x-3}{4}\right)$

$= 4\left(\frac{x-3}{4}\right) + 3$

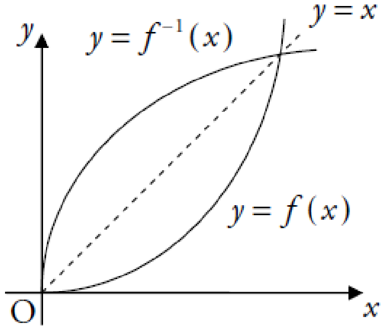
$= x - 3 + 3$

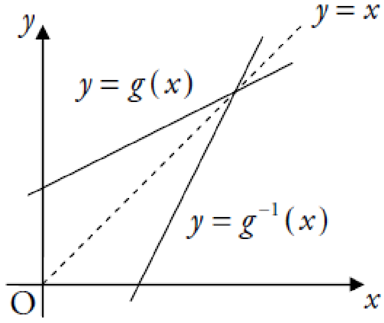
$= x$

Graphs of inverse functions

If we have the graph of a function, then we can find the graph of its inverse by reflecting in the line $y = x$

For example, the diagrams below show the graphs of two functions and their inverses.

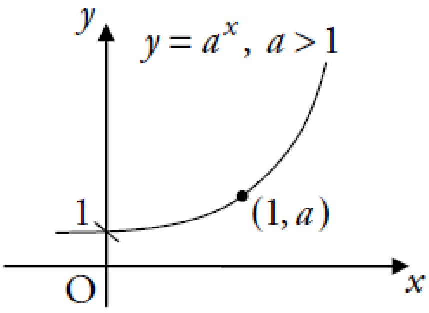


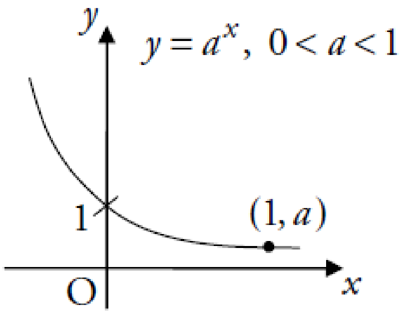


Two inverse functions we must know are exponential functions and logarithmic functions.

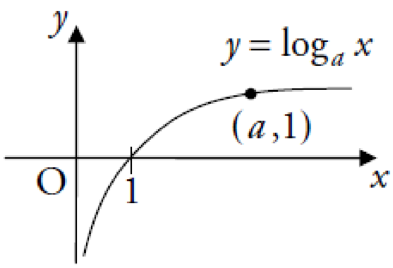
An exponential function is one in the form *f(x)* = a^x where a, x and *a* > 0

The graph of an exponential always passes through (0,1) and (1,a).





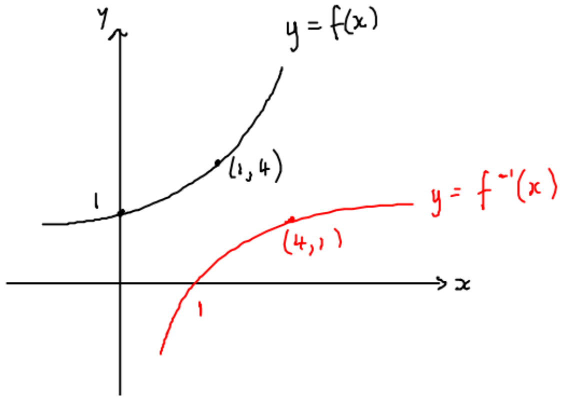
A logarithmic function is one in the form *f(x)* = $\log_a x$ where $a, x > 0$



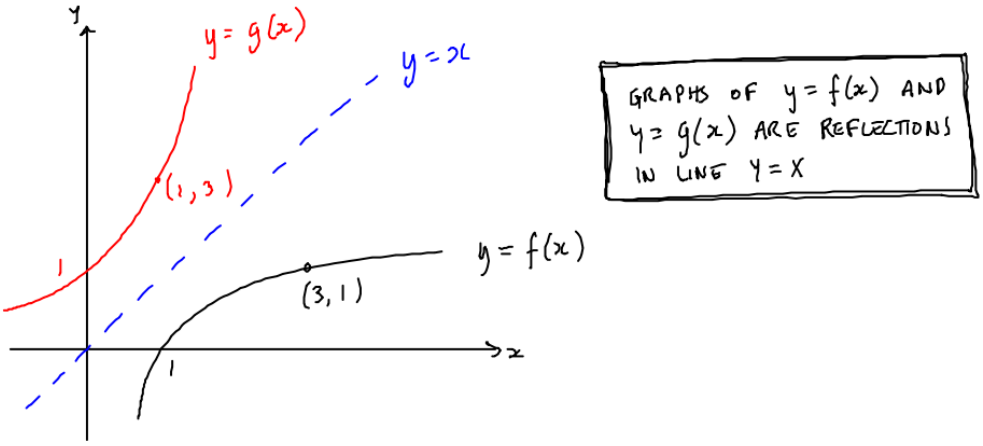
Log functions and exponential functions are inverses, so to find the graph of $y = \log_a x$ we reflect the graph of $y = a^x$ in the line $y = x$

Examples

F-10 If $f(x) = 4^x$, sketch the graph of $y = f(x)$ and $y = f^{-1}(x)$

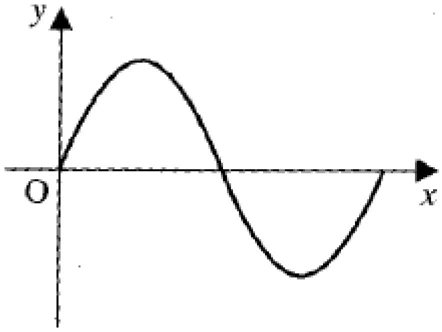


F-11 Sketch the graph of $f(x) = \log_3 x$ and $g(x) = 3^x$ and state why they are inverse functions

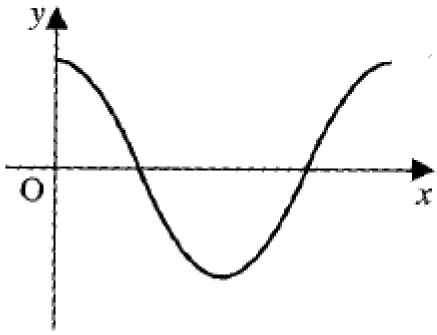


Graphs of standard functions

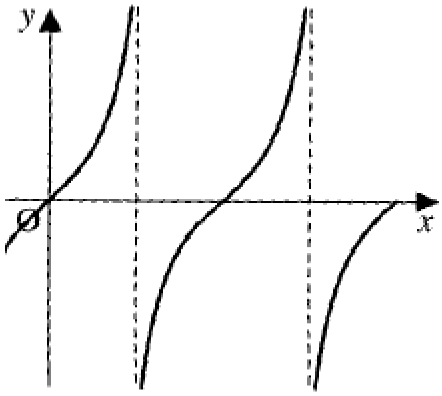
We should recognise the functions of the following graphs



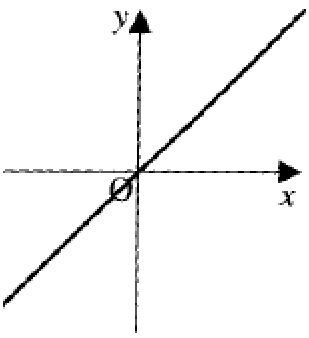
$y = sinx$



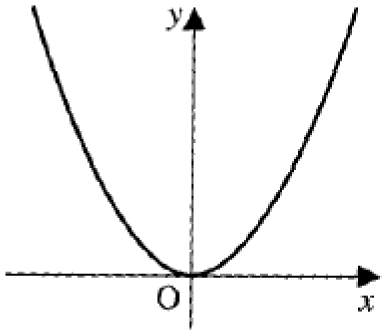
$y = cosx$



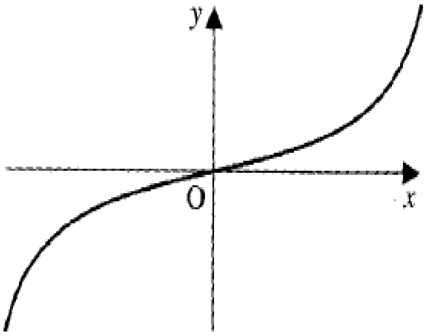
$y = tanx$



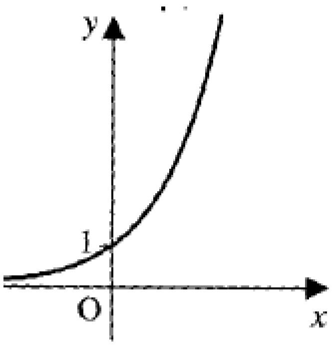
$y = x$



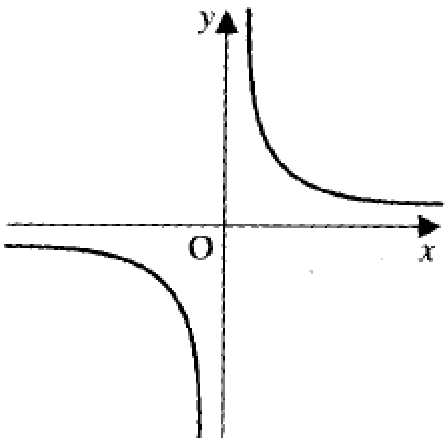
$y = x^2$



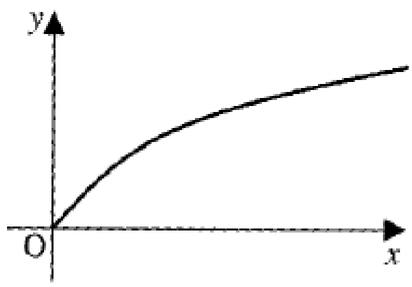
$y = x^3$



$y = a^x$



$y = \frac{1}{x}$



$y = \sqrt{x}$

Transformation of graphs

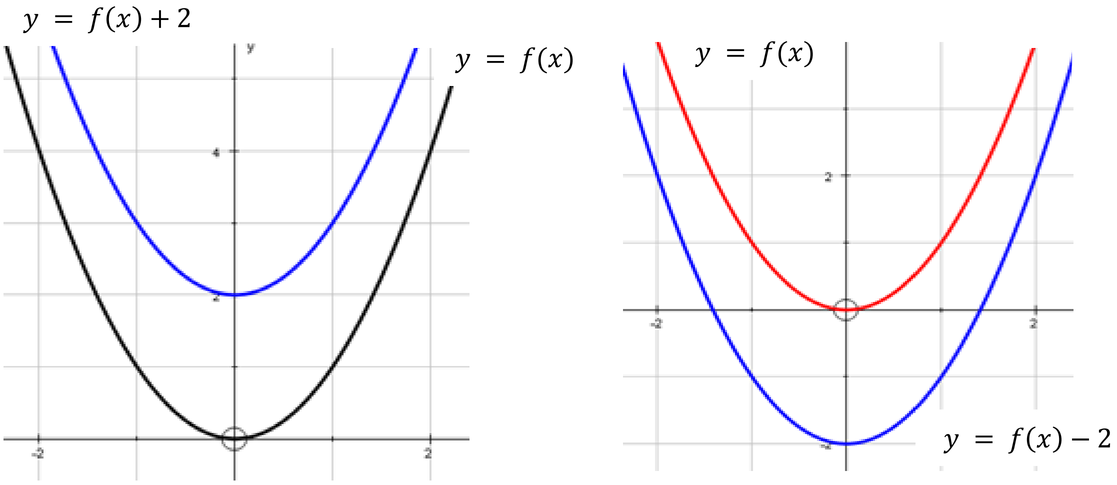
$f(x) + a$

The graph of $f(x) + a$ moves the graph $f(x)$ vertically (up or down the y axis).

* If $a > 0$ the graph moves up
* If $a < 0$ the graph moves down

Note: $y = f(x) + 2$ can be written as $y = 2 + f(x)$

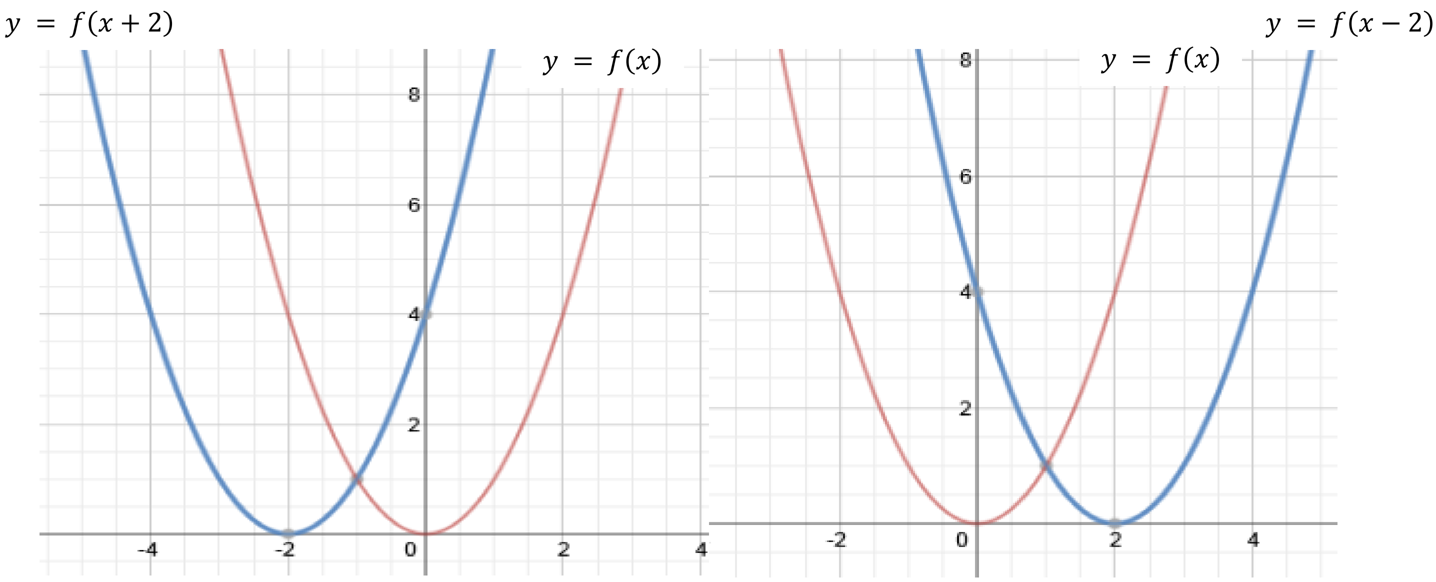
$y = f(x) - 2$ can be written as $y = -2 + f(x)$



$f(x + a)$

The graph of $f(x + a)$ moves the graph $f(x)$ horizontally (left or right along the x axis).

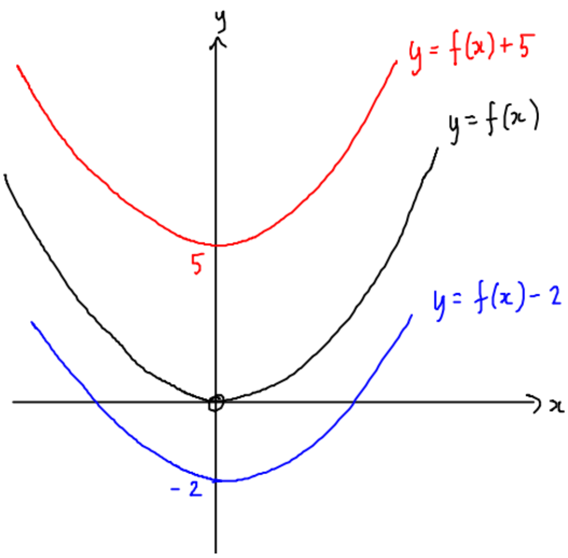
* If $a > 0$ the graph moves to the left
* If $a < 0$ the graph moves to the right

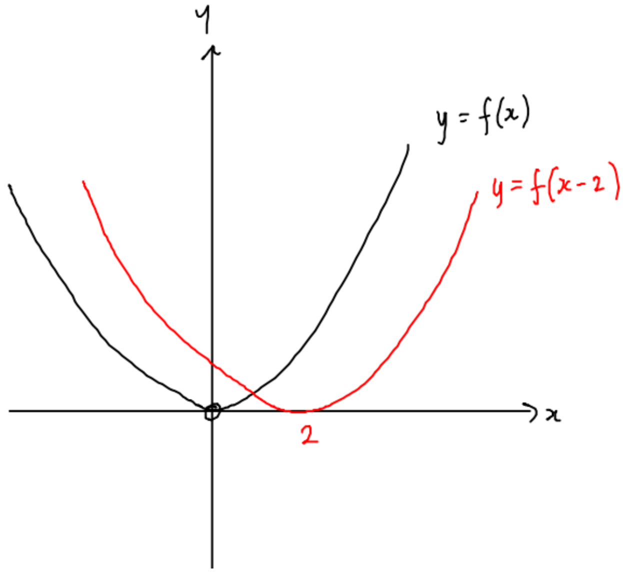


Examples

F-12 Sketch the graph of $f(x) = x^2$. On the same graph sketch the graph of

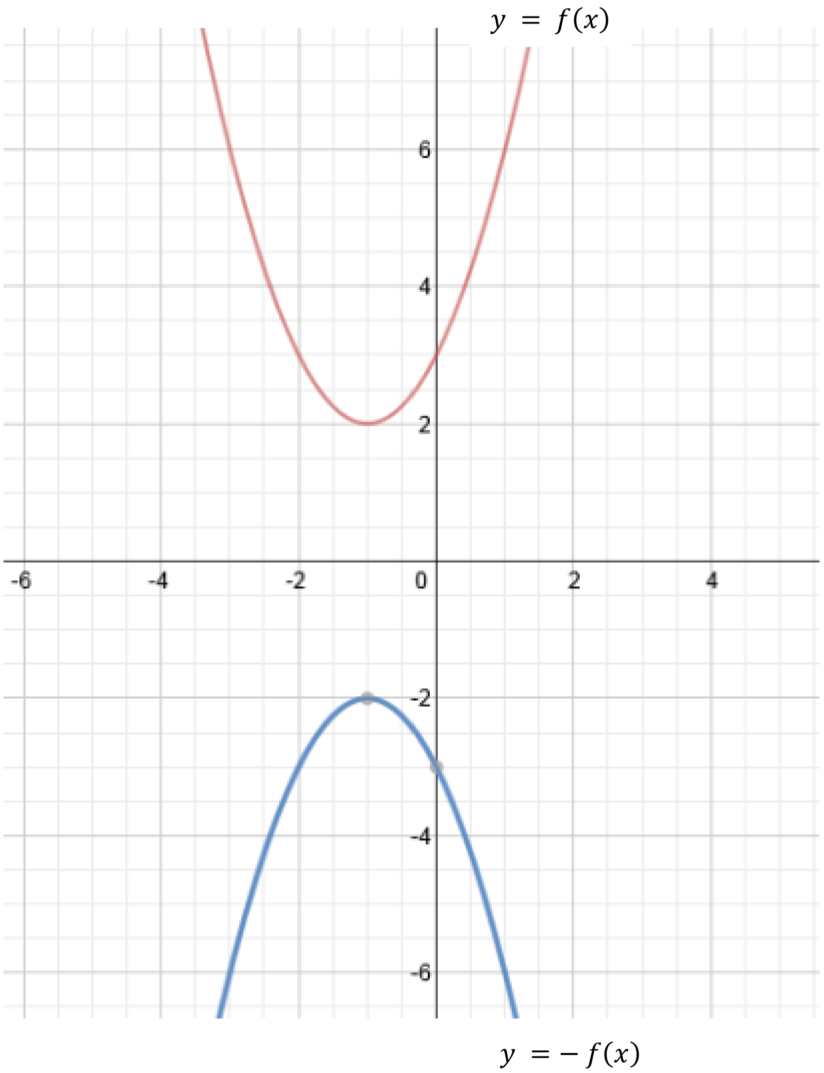
(i) $f(x) + 5$ (ii) $f(x) - 2$ (iii) $f(x - 2)$





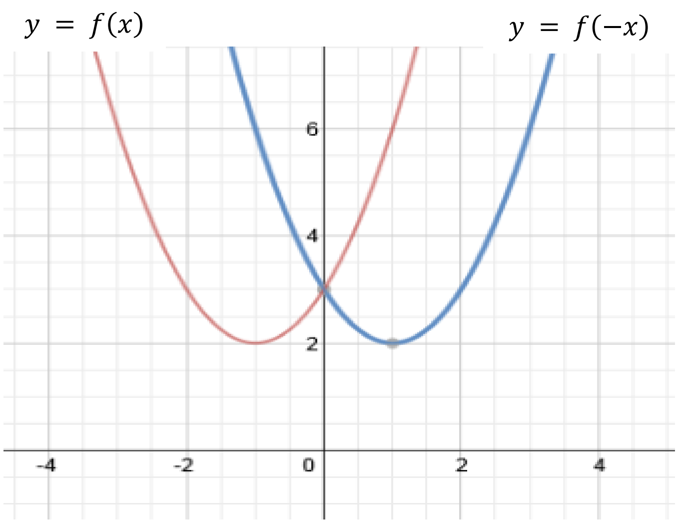
$-f(x)$

The graph of -$f(x)$ reflects the graph of $f(x)$ in the x-axis.



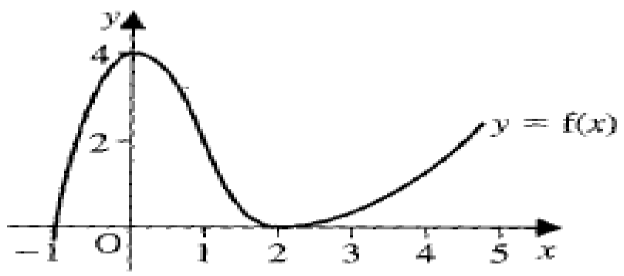
$f(-x)$

The graph of $f(-x)$ reflects the graph of $f(x)$ in the y-axis



Examples

F-13 Part of the graph $y = f(x)$ is shown.

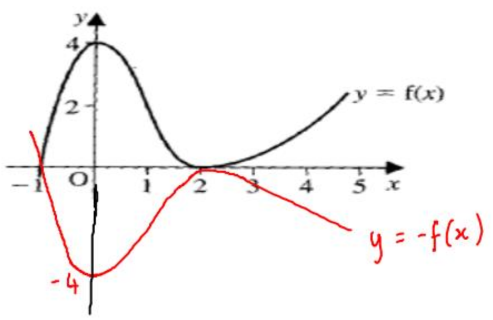


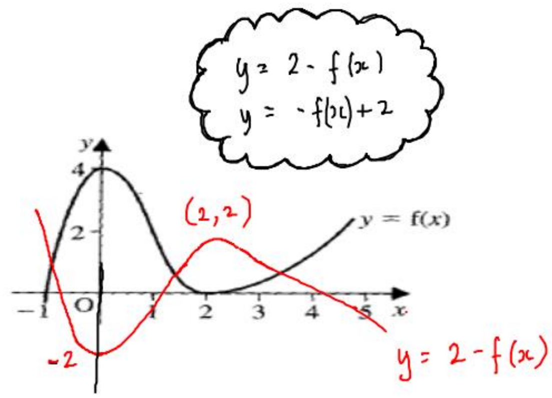
On the same diagram:

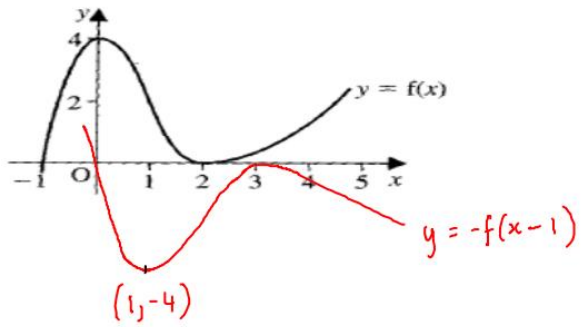
(i) Sketch the graph of $y = f(x)$ and $y = -f(x)$

(ii) Sketch the graph of $y = f(x)$ and $y = 2 - f(x)$

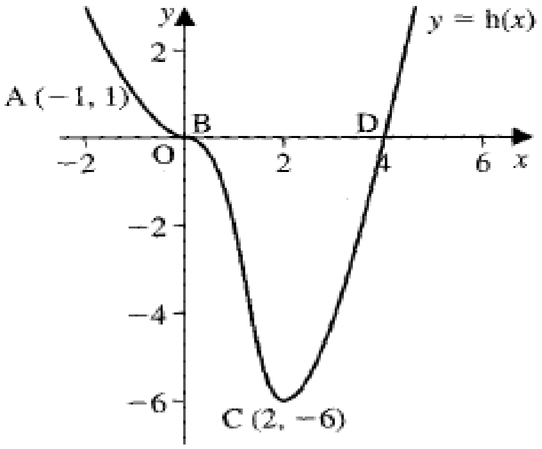
(iii) Sketch the graph of $y = f(x)$ and $y = -f(x-1)$







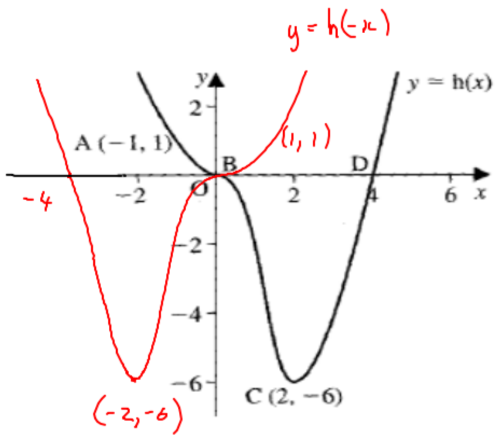
F-14 Part of the graph $y = h(x)$ is shown.

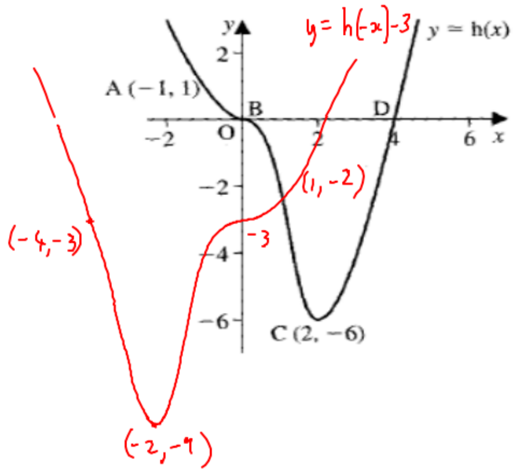


On the same diagram

(i) Sketch the graph of $y = h(x)$ and $y = h(-x)$

(ii) Sketch the graph of $y = h(x)$ and $y = h(-x) - 3$

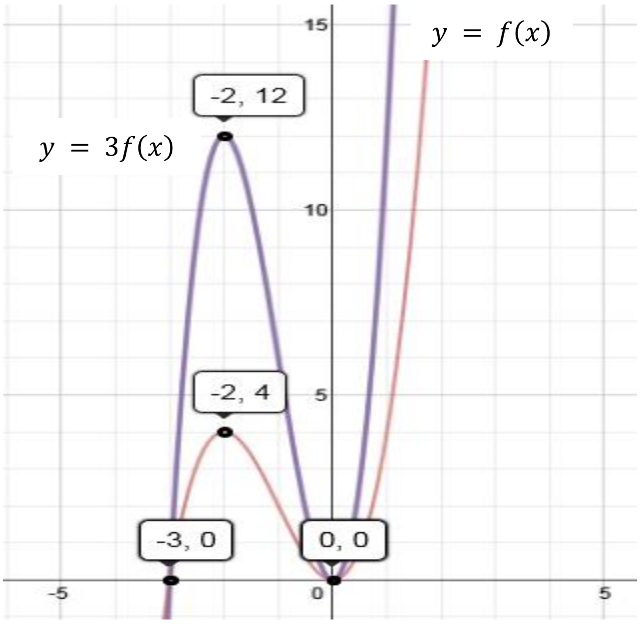


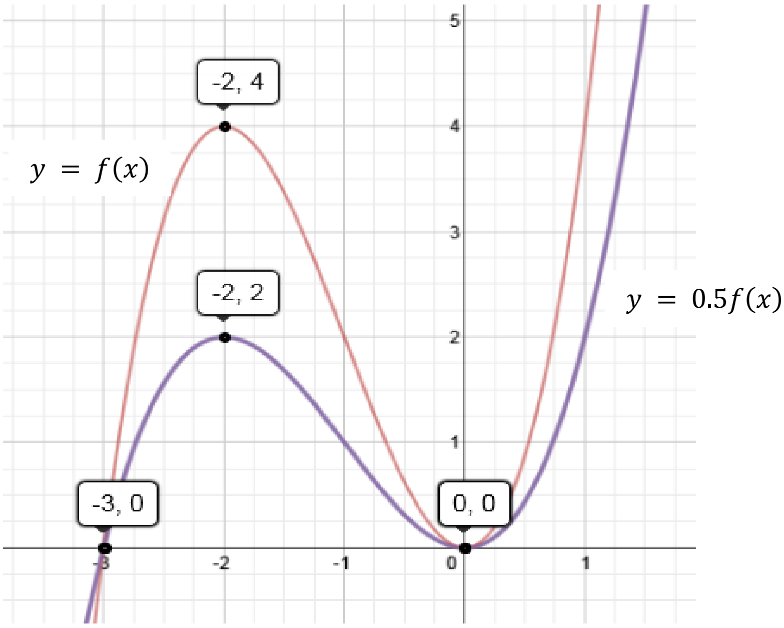


$kf(x)$

The graph of $kf(x)$ stretches or compresses the $f(x)$ vertically. Think $y = 3sinx$

* If $k > 1$ the graph stretches
* If $k < 1$ the graph compresses

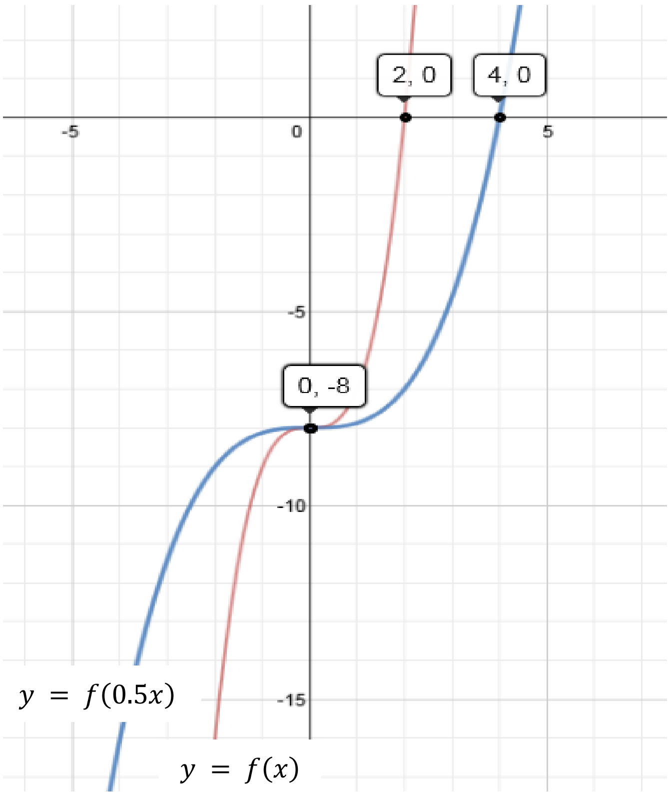


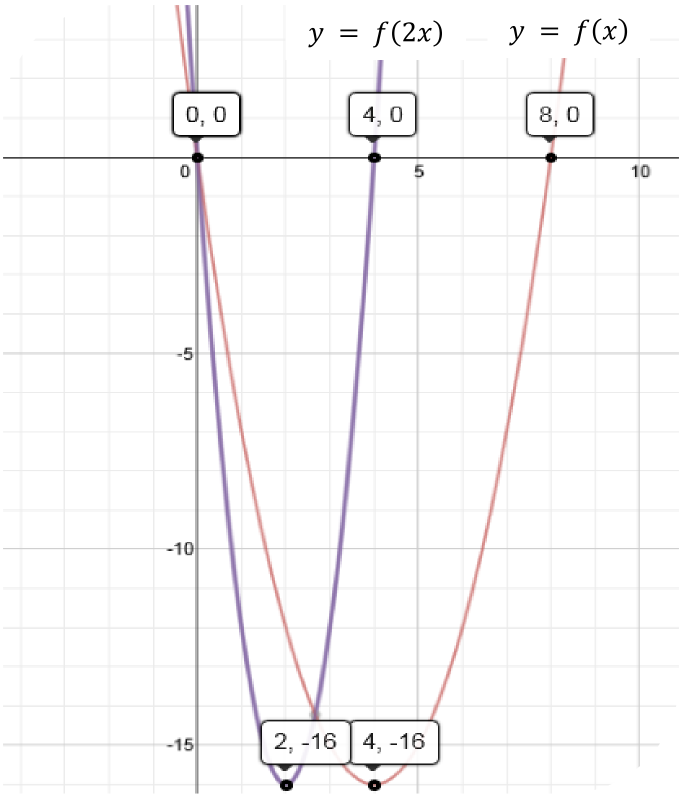


$f(kx)$

The graph of $f(kx)$ stretches or compresses the $f(x)$ horizontally. Think $y = sin3x$

* If $k > 1$ the graph compresses
* If $k < 1$ the graph stretches

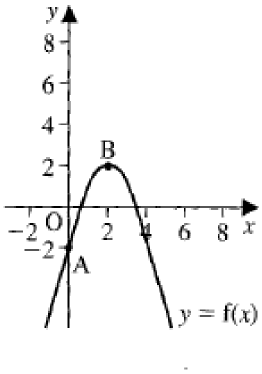


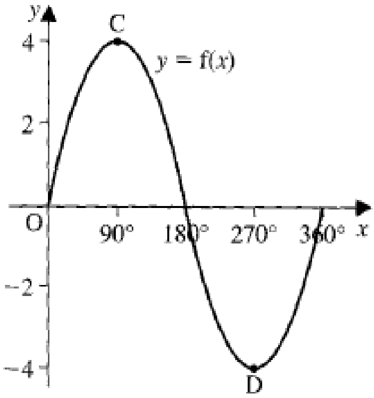


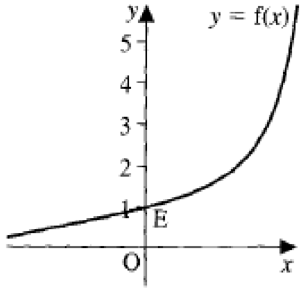
Examples

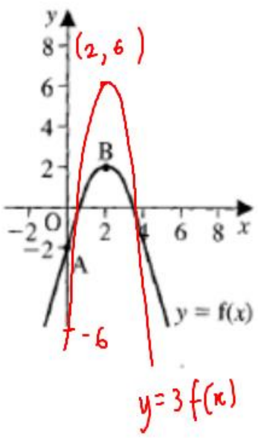
F-15 Copy each graph and, on the same diagram, sketch the graph of the given related function. Annotate the sketches with the coordinates of the images of A, B, C, D and E.

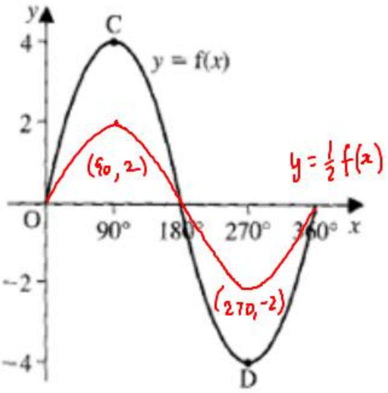
(a) $y = 3f(x)$ **(b)** $y = \frac{1}{2}f(x)$ **(c)** $y = 2f(x)$

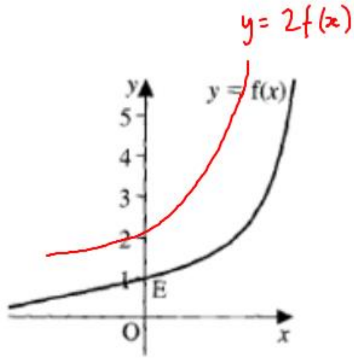






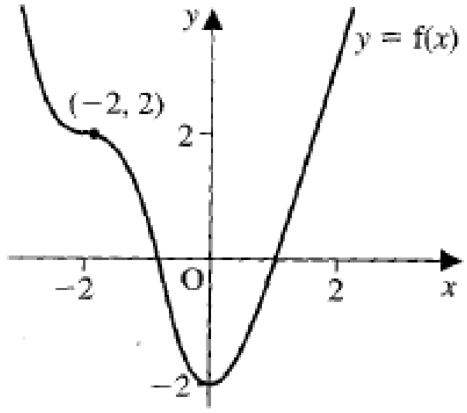


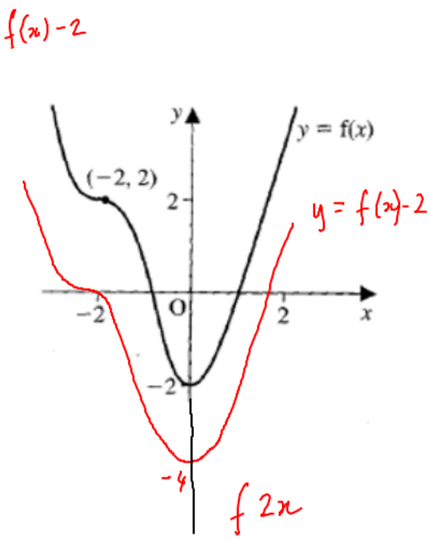


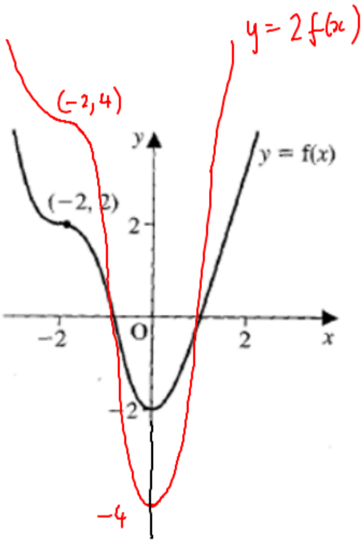


F-16 The graph of $y = f(x)$ is shown. On separate diagrams sketch the graphs of

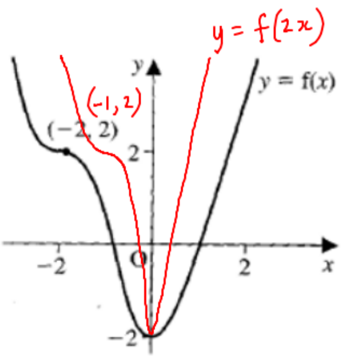
(a) $y = f(x) - 2$ (b) $y = 2f(x)$ (c) $y = f(2x)$ (d) $y = -2f(x)$



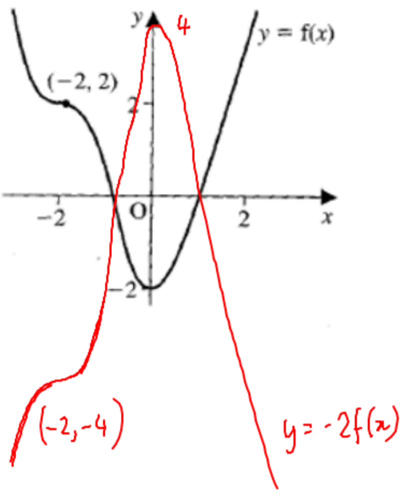




f 2x



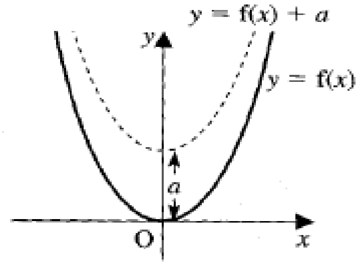
-2f(x)



Summary of Transformation of Graphs

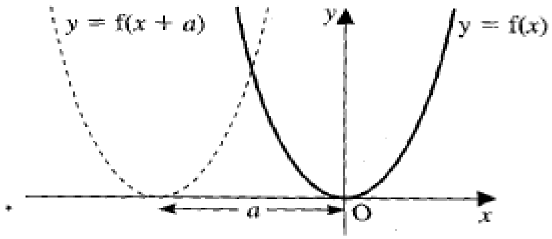
① To obtain the graph of $y = \text{f}(x) + a$, **slide** $y = \text{f}(x)$ **vertically**

upwards for $a > 0$
downwards for $a < 0$.

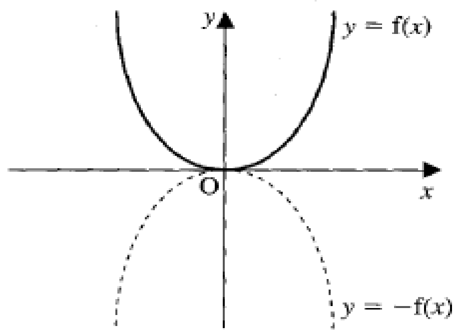


② To obtain the graph of $y = \text{f}(x + a)$, **slide** $y = \text{f}(x)$ **horizontally**

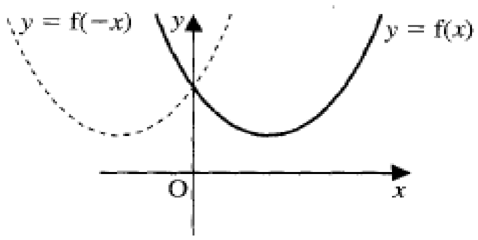
to the left for $a > 0$
to the right for $a < 0$.



③ To obtain the graph of $y = -\text{f}(x)$, **reflect** $y = \text{f}(x)$ **in the *x*-axis**.

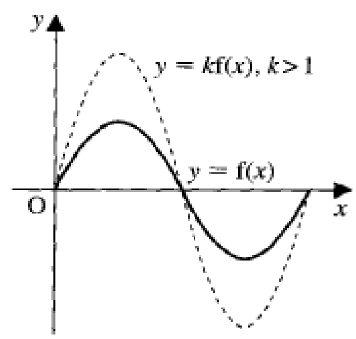


④ To obtain the graph of $y = \text{f}(-x)$, **reflect** $y = \text{f}(x)$ **in the *y*-axis**.



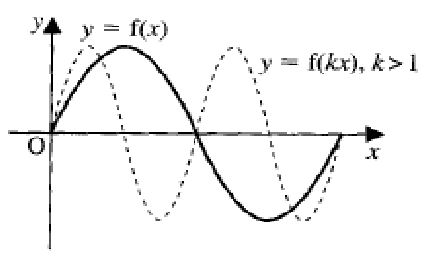
⑤ To obtain the graph of $y = k\text{f}(x)$, **stretch** or **compress** $y = \text{f}(x)$ **vertically** by a factor of k:

stretch for $k > 1$
compress for $k < 1$.



⑥ To obtain the graph of $y = \text{f}(kx)$, **stretch** or **compress** $y = \text{f}(x)$ **horizontally** by a factor of k:

compress for $k > 1$
stretch for $k < 1$.



Summary of Functions

functions & graphs

COMPOSITE FUNCTIONS:

(substituting one function into another)

$f(x) = x^2 \quad g(x) = x + 2$

$f(g(x)) = f(x+2) = (x+2)^2$

$g(f(x)) = g(x^2) = x^2 + 2$

$f(g(x)) \neq g(f(x))$

just work inside out!

NOTE: If $f(g(x)) = x$ then $f(x)$ & $g(x)$ are inverse functions

RESTRICTED DOMAINS:

Denominator of fraction $\neq 0$

Number under root ≥ 0

INVERSE FUNCTIONS:

1. Set $f(x) = y$
2. rearrange to $x =$
3. substitute x and y

$f(x) = 3x^2 + 2$

$y = 3x^2 + 2$

$3x^2 = y - 2$

$x^2 = \frac{y-2}{3}$

$x = \sqrt{\frac{y-2}{3}}$

$f^{-1}(x) = \sqrt{\frac{x-2}{3}}$

NOTE: Graphs of inverse functions are reflected in line $y = x$