HIGHER MATHS

Exponentials & Logarithms

Notes with Examples

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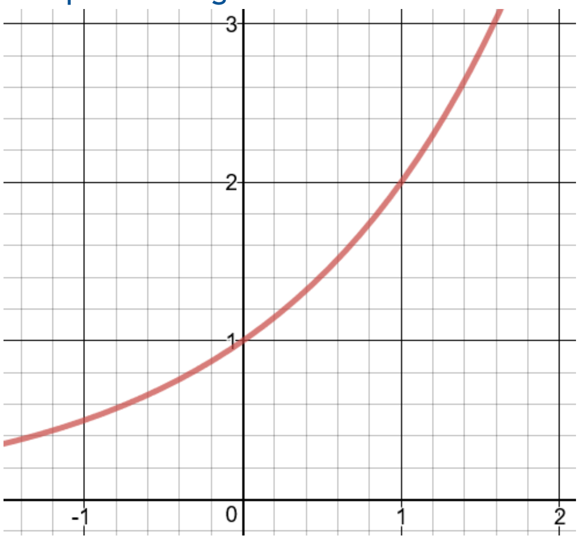
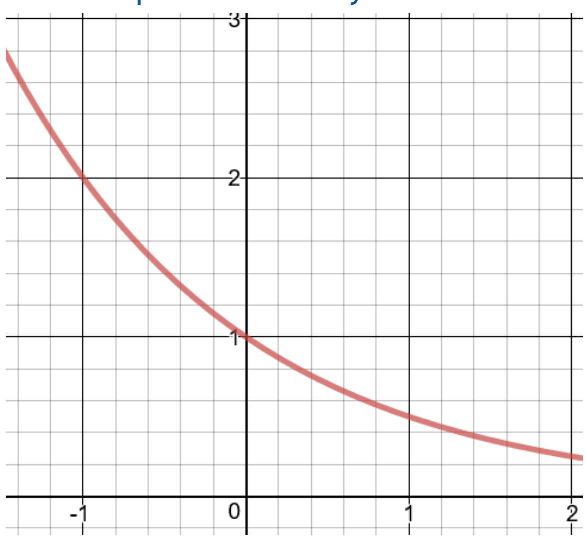
Exponential Growth & Decay Graphs

You should be familiar with the graphs of exponential functions. The graph of the exponential function $y = a^x$ will pass through $(0,1)$ and $(1, a)$ where a is referred to as the base.

Exponential growth when $a > 1$



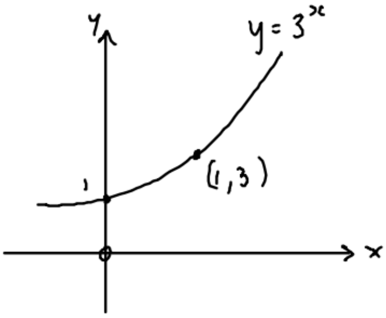
Exponential decay when $a < 1$

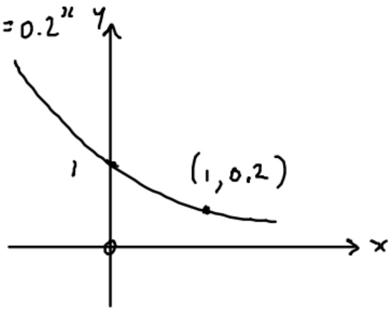


Examples

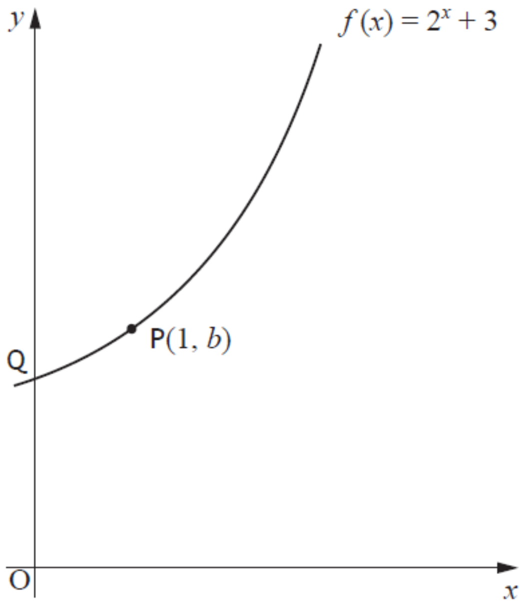
LE-01 Sketch the graph of

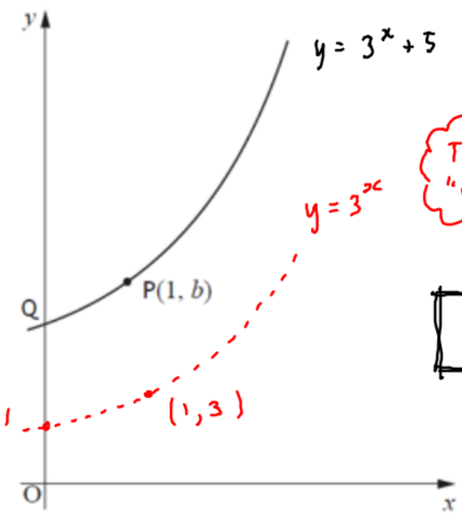
(a) $y = 3^x$ (b) $y = 0.2^x$





LE-02 The graph below shows the function $f(x) = 3^x + 5$. State the coordinates of P and Q.





THINK ABOUT THE "ORIGINAL" GRAPH

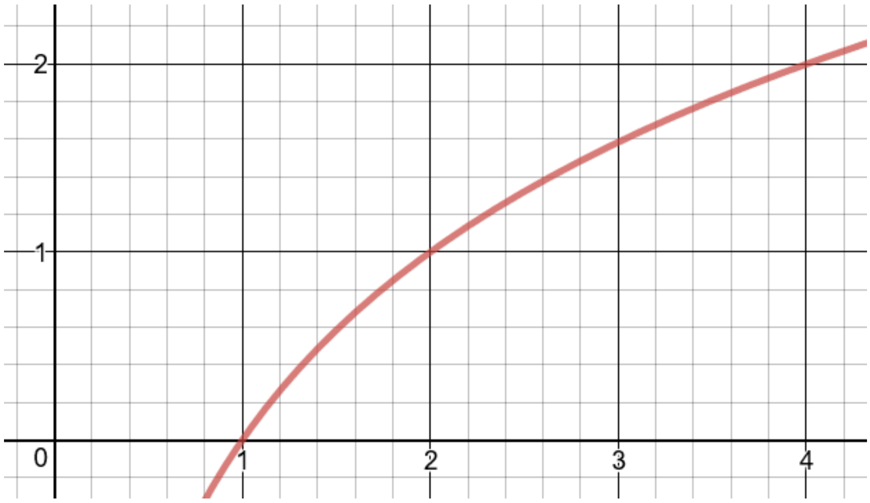
+5

$\boxed{P(1,8) \quad Q(0,6)}$

Logarithms

A logarithm is the inverse function of an exponential function and we have already seen the graph of a logarithmic function previously in the course.

The graph of the logarithmic function $y = \log_a x$ will pass through $(1,0)$ and $(a, 1)$ where a is referred to as the base.



To work with logarithms you need to understand what they mean.
In general, $\log_a x$ means "a to the power of what gives the value of x". This means we can change between logarithmic and exponential form using:

$$\boxed{y = \log_a x \quad \leftrightarrow \quad a^y = x}$$

Examples

LE-03 Evaluate

(a) $\log_3 9$

$3^x = 9$

$x = 2$

$\boxed{\log_3 9 = 2}$

(b) $\log_5 125$

$5^x = 125$

$x = 3$

$\boxed{\log_5 125 = 3}$

(c) $\log_2 64$

$2^x = 64$

$x = 6$

$\boxed{\log_2 64 = 6}$

(d) $\log_{16} 4$

$16^x = 4$

$x = 1/2$

$\boxed{\log_{16} 4 = 1/2}$

LE-04 Calculate the value of x

(a) $\log_x 16 = 2$

$x^2 = 16$

$\boxed{x = 4}$

(b) $\log_2 x = 6$

$2^6 = x$

$\boxed{x = 64}$

(c) $\log_7 7 = x$

$7^x = 7$

$\boxed{x = 1}$

Laws of Logarithms

There are several laws of Logarithms you must remember.

Addition of Logs

When two logarithms **with the same base** are added together, the terms can be combined by multiplying the arguments.

$$\boxed{\log_a x + \log_a y = \log_a(xy)}$$

Subtraction of Logs

When two logarithms **with the same base** are subtracted, the terms can be combined by dividing the arguments.

$$\boxed{\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)}$$

Powers of Logs

When dealing with a logarithm whose argument has a power, we can move the power to the front of the logarithm as a multiplier. This also works in reverse.

$$\boxed{\log_a x^n = n\log_a x}$$

Special cases

There are two special logarithms we should remember.

$$\boxed{\log_a 1 = 0} \qquad \boxed{\log_a a = 1}$$

Examples

LE-05 Evaluate $\log_8 2 + \log_8 32$

$$\log_8 2 + \log_8 32$$
$$= \log_8(2\times 32)$$
$$= \log_8 64$$

8 to the power of what gives 64?

$$= 2$$

LE-06 Evaluate $\log_5 250 - \log_5 2$

$$\log_5 250 - \log_5 2$$
$$= \log_5\left(\frac{250}{2}\right)$$
$$= \log_5 125$$
$$= 3$$

LE-07 Evaluate $2\log_2 4 - 3\log_2 2$

$$2\log_2 4 - 3\log_2 2$$
$$= \log_2 4^2 - \log_2 2^3$$
$$= \log_2 16 - \log_2 8$$
$$= \log_2\left(\frac{16}{8}\right)$$
$$= \log_2 2$$
$$= 1$$

LE-08 Evaluate $5\log_8 2 + \log_8 4 - \log_8 16$

$$5\log_8 2 + \log_8 4 - \log_8 16$$
$$= \log_8 2^5 + \log_8 4 - \log_8 16$$
$$= \log_8\left(\frac{32 \times 4}{16}\right)$$

You can do this in stages if you want

$$= \log_8 8$$
$$= 1$$

Exponential and Logarithmic Functions with base e

The exponential function e^x, base e is a very special function. We use this function in most exponential calculations and $e = 2.718.....$

The inverse function to e^x is called the natural logarithm and can be denoted by either $\log_e x$ or $\ln x$.

Examples

LE-09 Evaluate

(a) e^4 (b) $e^{7.2}$

e^4

$= 54.598$

≈ 54.6

$e^{7.2}$

$= 1339.43$

$= 1339.4$

LE-10 The population of sea monkeys in a tank increases exponentially and can be expressed by the formula

$$P(t) = 2e^{0.6t},$$

where $P(t)$ is the number of sea monkey after t weeks.

(a) How many sea monkeys are in the tank to start initially?

(b) How many sea monkeys will there be after 5 weeks?

(a) $P(t) = 2e^{0.6t}$

$P(0) = 2e^{0.6(0)}$

$\boxed{P(0) = 2}$

(b) $P(5) = 2e^{0.6(5)}$

$= 2e^3$

$= 40.17$

$\boxed{P(5) = 40}$

LE-11 The number of bacteria in a petri dish is given by the formula $B(t) = 20e^{1.2t}$, where t is the time in hours.

(a) How many bacteria are there at time zero?

(b) How long will it take for the number of bacteria to triple?

(a)

$$B(t) = 20e^{1.2t}$$

$$B(0) = 20e^{1.2(0)}$$

$$\boxed{B(0) = 20}$$

(b) TO TRIPLE $B(t) = 60$

$$60 = 20e^{1.2t}$$

$$e^{1.2t} = \frac{60}{20}$$

$$e^{1.2t} = 3$$

$$1.2t = \ln 3$$

$$t = \frac{\ln 3}{1.2}$$

$$t = 0.92$$

IT WILL TAKE 0.92 HOURS FOR THE BACTERIA TO TRIPLE

LE-12 The formula $A = A_0e^{-kt}$ gives the amount of radioactive substance after time t minutes. After 4 minutes 50g is reduced to 45g.

(a) Find the value of k to two significant figures?

(b) How long does it take for the substance to reduce to half its original weight?

(a)

$$A = A_0 e^{-kt}$$

$A_0 = 50$ $A = 45$ $t = 4$

$$45 = 50e^{-4k}$$

$$e^{-4k} = \frac{45}{50}$$

$$-4k = \ln(0.9)$$

$$k = \frac{\ln(0.9)}{-4}$$

$$= 0.02634$$

$$\boxed{k = 0.026}$$

(b) FOR HALF WEIGHT $A = \frac{1}{2}A_0$

$$A = A_0 e^{-0.026t}$$

$$\frac{1}{2}A_0 = A_0 e^{-0.026t}$$

$$e^{-0.026t} = \frac{1}{2}$$

$$-0.026t = \ln\left(\frac{1}{2}\right)$$

$$t = \frac{\ln\left(\frac{1}{2}\right)}{-0.026}$$

$$t = 26.7$$

IT TAKES 26.7 MINUTES TO HALF ITS WEIGHT

Exponential and Logarithmic Equations

Using the Laws of Logarithms and the connection between logarithms and exponentials, we can solve equations.

Examples

LE-13 Solve

(a) $5^x = 35$ (b) $5^x = \frac{1}{10}$ (c) $e^x = 0.28$

(a)

$$5^x = 35$$
$$\log 5^x = \log 35$$
$$x \log 5 = \log 35$$
$$x = \frac{\log 35}{\log 5}$$
$$\boxed{x = 2.2}$$

(b)

$$5^x = \frac{1}{10}$$
$$\log 5^x = \log \frac{1}{10}$$
$$x \log 5 = \log \frac{1}{10}$$
$$x = \frac{\log \frac{1}{10}}{\log 5}$$
$$\boxed{x = -1.43}$$

(c)

$$e^x = 0.28$$
$$\ln e^x = \ln 0.28$$
$$x \ln e = \ln 0.28$$
$$\boxed{x = -1.27}$$

LE-14 Solve for $x > 0$, $\log_a x + \log_a 2 = \log_a 10$

$$\log_a x + \log_a 2 = \log_a 10$$
$$\log_a x = \log_a 10 - \log_a 2$$
$$\log_a x = \log_a \left(\frac{10}{2}\right)$$
$$\log_a x = \log_a 5$$
$$\boxed{x = 5}$$

LE-15 Solve for $x > -1$, $\log_5(x+1) + \log_5(x-3) = 1$

$$\log_5 (x+1) + \log_5 (x-3) = 1$$
$$\log_5 (x+1)(x-3) = \log_5 5$$
$$x^2 - 3x + x - 3 = 5$$
$$x^2 - 2x - 8 = 0$$
$$(x+2)(x-4) = 0$$
$$x = -2 \quad \boxed{x = 4}$$

LE-16 Solve for $x > 4$, $\log_e(x-4) + \log_e(2x+5) = \log_e 3x$

$$\log_e(x-4) + \log_e(2x+5) = \log_e 3x$$

$$\log_e (x-4)(2x+5) = \log_e 3x$$

$$2x^2 + 5x - 8x - 20 = 3x$$

$$2x^2 - 6x - 20 = 0$$

$$(2x-10)(x+2) = 0$$

$$\boxed{x=5} \quad x = -2$$

Experimental Data

Analysing data from experiments may result in a relationship that indicates exponential growth. To fully analyse this data and the relationship the graph must be in the form of a straight line. We do this by using logarithms.

Data in the form $y = kx^n$

By taking logs of both sides of the equation and simplifying we can write in the form of the equation of a straight line.

$$y = kx^n$$
$$\log y = \log kx^n$$
$$\log y = \log k + \log x^n$$
$$\log y = \log k + n \log x$$
$$\log y = n \log x + \log k$$

This resembles $\quad y = mx + c$

where $m = n$ and $c = \log k$.

Data in the form $y = ab^x$

By taking logs of both sides of the equation and simplifying we can write in the form of the equation of a straight line.

$$y = ab^x$$
$$\log y = \log ab^x$$
$$\log y = \log a + \log b^x$$
$$\log y = \log a + x \log b$$
$$\log y = x \log b + \log a$$

This resembles $\quad y = mx + c$

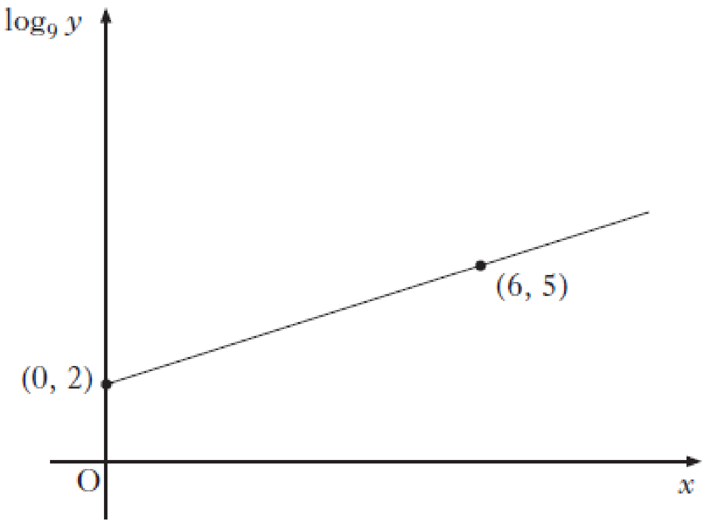
where $m = \log b$ and $c = \log a$.

Examples

LE-17 Two variables, x and y, are related by the equation

$$y = ka^x.$$

When $\log_9 y$ is plotted against x, a straight line passing through the points (0, 2) and (6, 5) is obtained, as shown in the diagram.



use $\log_9$ as its in the graph

$y = ka^x$

$\log_9 y = \log_9 ka^x$

$= \log_9 k + \log_9 a^x$

$= \log_9 k + x \log_9 a$

$\log_9 y = x\log_9 a + \log_9 k$

$m = \log_9 a$

$c = \log_9 k$

Find the values of k and a.

$m = \frac{V}{H} = \frac{3}{6} = \frac{1}{2}$

$\log_9 a = \frac{1}{2}$

$a = 9^{1/2}$

$= \sqrt{9}$

$\boxed{a = 3}$

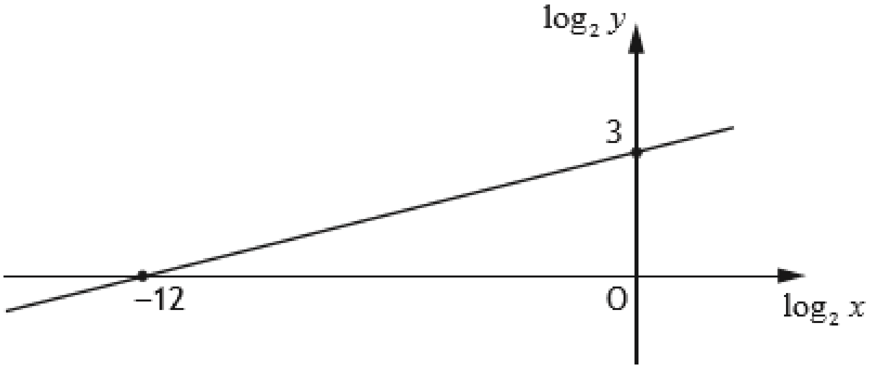
$\log_9 k = 2$

$k = 9^2$

$\boxed{k = 81}$

LE-18 Two variables, x and y, are connected by the equation $y = kx^n$.

The graph of $\log_2 y$ against $\log_2 x$ is a straight line as shown.



Find the values of k and n.

$y = kx^n$

$\log_2 y = \log_2 kx^n$

$\log_2 y = \log_2 k + \log_2 x^n$

$\log_2 y = \log_2 k + n\log_2 x$

$\log_2 y = n\log_2 x + \log_2 k$

$m = n \quad c = \log_2 k$

Use $\log_2$ as its in the graph

$n = m$

$= \frac{V}{H}$

$= \frac{3}{12}$

$\boxed{n = \frac{1}{4}}$

$c = \log_2 k$

$\log_2 k = 3$

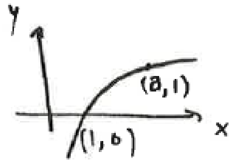
$k = 2^3$

$\boxed{k = 8}$

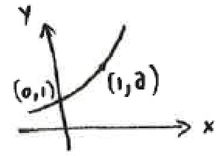
Summary

Logs & Exponentials

GRAPHS:



$y = \log_a x$



$y = a^x$

QUESTION
can you move these graphs or get back to the original graph if given a "moved" graph?

WHAT IS A LOG?:

$\log_3 9$ means "3 to the power of what equals 9?"

BASE

so $\log_3 9 = 2$

that means we can swap between logs & exponentials

$$\log_a y = x \iff y = a^x$$

BASE

RULES OF LOGS: these work both ways!

$$\log_a x + \log_a y = \log_a xy$$

$$\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$$

MUST BE SAME BASE

$$\log_a x^n = n\log_a x$$

$$\log_a 1 = 0$$

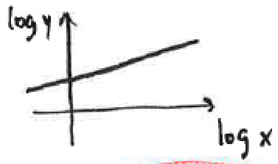
$$\log_a a = 1$$

You must remember these!

EXPERIMENTAL DATA:

this should always be a straight line

① Plot $\log y$ vs $\log x$



$y = kx^n$

$\log y = \log kx^n$

$= \log k + \log x^n$

$= \log k + n\log x$

$= n\log x + \log k$

so $\log k = c$ (y int)
$n =$ gradient

② Plot $\log y$ vs x

$y = ab^x$

$\log y = \log ab^x$

$= \log a + \log b^x$

$= \log a + x\log b$

$= x\log b + \log a$

So $\log a = c$ (y-int)
$\log b =$ gradient