Trigonometric Equations

Most trigonometric equations can be divided into one of three types:

TYPE 1: Equations involving a trigonometric function squared but no other trigonometric function.

Examples $4\sin^2 x + 5 = 6$, $3\tan^2 x - 9 = 0$

TYPE 2: Equations involving 2x, 3x, etc. but no other trigonometric function.

Examples $3\sin 2x - 1 = 1$, $\sqrt{3}\tan(3x - 30) + 2 = 1$

TYPE 3: Equations involving 2x and another trigonometric function i.e. equations involving the double angle formulae.

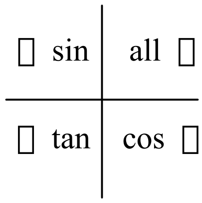
Examples $4\sin 2x - 2\cos x = 0$, $\cos 2x - 1 = 3\cos x$

TYPE 1:

Example 1 Solve $4\sin^2 x + 5 = 6$ $0 \le x \le 360$

Solution:

$$4\sin^2 x + 5 = 6$$
$$4\sin^2 x = 1$$
$$\sin^2 x = \tfrac{1}{4}$$
$$\sin x = \tfrac{1}{2}, -\tfrac{1}{2}$$

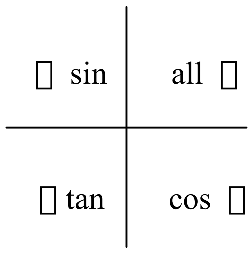


$$x = 30^0, 150^0, 210^0, 330^0$$

Example 2 Solve $3\tan^2 x - 9 = 0$ $0 \le x \le 360$

Solution:

$$3\tan^2 x - 9 = 0$$
$$3\tan^2 x = 9$$
$$\tan^2 x = 3$$
$$\tan x = \sqrt{3}, -\sqrt{3}$$



$$x = 60^0, 120^0, 240^0, 300^0$$

TYPE 2:

Example 1 Solve $3\sin 2x - 1 = 1 \quad 0 \le x \le 360$

(Since question involves 2x change range to $0 \le x \le 720$)

Solution:

$$3\sin 2x - 1 = 1$$
$$3\sin 2x = 2$$
$$\sin 2x = \tfrac{2}{3}$$
$$2x = 41.8^0, 138.2^0, 360^0 + 41.8^0, 360^0 + 138.2^0$$
$$x = 20.9^0, \ 69.1^0, 200.9^0, 249.1^0$$

□ sin	all □
tan	cos

Example 2 Solve $\sqrt{3}\tan(3x - 30) + 2 = 1 \quad 0 \le x \le 180$

(Since question involves 3x change range to $0 \le x \le 540$)

Solution:

$$\sqrt{3}\tan(3x - 30) + 2 = 1$$
$$\sqrt{3}\tan(3x - 30) = -1$$
$$\tan(3x - 30) = -\tfrac{1}{\sqrt{3}}$$
$$3x - 30 = 150^0, 330^0, 360^0 + 150^0, 360^0 + 330^0$$
$$3x - 30 = 150^0, 330^0, 510^0, 690^0 \text{(too big)}$$
$$3x = 180^0, 360^0, 540^0$$
$$x = 60^0, 120^0, 180^0$$

□ sin	all
tan	cos □

TYPE 3:

Example 1 Solve $4\sin 2x - 2\cos x = 0 \quad 0 \le x \le 360$

Solution: (Use the formula $\sin 2x = 2\sin x \cos x$)

$$4\sin 2x - 2\cos x = 0$$
$$4(2\sin x \cos x) - 2\cos x = 0$$
$$8\sin x \cos x - 2\cos x = 0$$
$$2\cos x(4\sin x - 1) = 0$$

$2\cos x = 0$ **or** $4\sin x - 1 = 0$

$\cos x = 0$

using graph: $x = 90^0, 270^0$

$4\sin x = 1$

$\sin x = \tfrac{1}{4}$

$x = 14.5^0, 165.5^0$

□ sin	all □
tan	cos

Example 2 Solve $\cos 2x - 1 = 3\cos x \quad 0 \le x \le 360$

Solution: (Use the formula $\cos 2x = 2\cos^2 x - 1$)

$$\cos 2x - 1 = 3\cos x$$
$$2\cos^2 x - 1 - 1 = 3\cos x$$
$$2\cos^2 x - 3\cos x - 2 = 0$$
$$(2\cos x + 1)(\cos x - 2) = 0$$

□ sin	all
□ tan	cos

$2\cos x + 1 = 0$ **or** $\cos x - 2 = 0$

$2\cos x = -1$ $\qquad$ $\cos x = 2$

$\cos x = -\frac{1}{2}$ $\qquad$ no solutions

$x = 120^0, 240^0$

NOTE: If equation involves **cos 2x** and **cos x** use the formula $\mathbf{\cos 2x = 2\cos^2 x - 1}$
If equation involves **cos 2x** and **sin x** use the formula $\mathbf{\cos 2x = 1 - 2\sin^2 x}$

Questions

1. Solve the following equations

(a) $3\tan^2 x - 1 = 0 \qquad 0 \le x \le 360$

(b) $2\cos 2x + 3 = 2 \qquad 0 \le x \le 360$

(c) $4\sin x - 3\sin 2x = 0 \qquad 0 \le x \le 360$

(d) $2\cos 2x = 1 - \cos x \qquad 0 \le x \le 360$

(e) $4\cos^2 x - 1 = 2 \qquad 0 \le x \le 2\pi$

(f) $5\tan(2x - 40) + 1 = 6 \qquad 0 \le x \le 360$

(g) $2\sin 2x + \sqrt{3} = 0 \qquad 0 \le x \le 2\pi$

(h) $3\sin 2x - 3\cos x = 0 \qquad 0 \le x \le 360$

(i) $\cos 2x + 5 = 4\sin x \qquad 0 \le x \le 360$

(j) $4\tan 3x + 5 = 1 \qquad 0 \le x \le \pi$

(k) $2\cos(2x + 80) = 1 \qquad 0 \le x \le 180$

(l) $6\sin^2 x + 5 = 8 \qquad 0 \le x \le 2\pi$

(m) $5\sin 2x - 6\sin x = 0 \qquad 0 \le x \le 360$

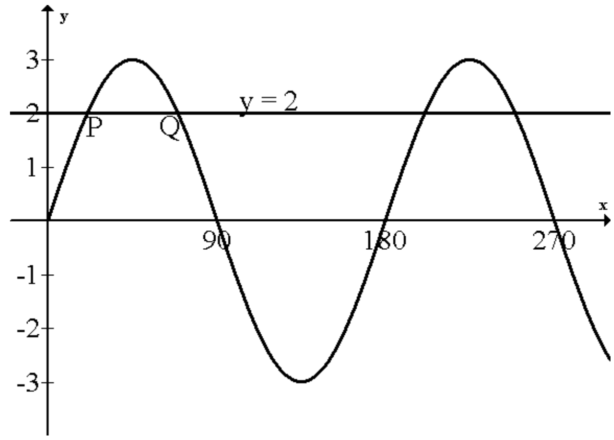
(n) $3\cos 2x + \cos x = -1 \qquad 0 \le x \le 360$

2. (a) Show that $2\cos 2x - \cos^2 x = 1 - 3\sin^2 x$

(b) Hence solve the equation $2\cos 2x - \cos^2 x = 2\sin x \quad 0 \le x \le 90$

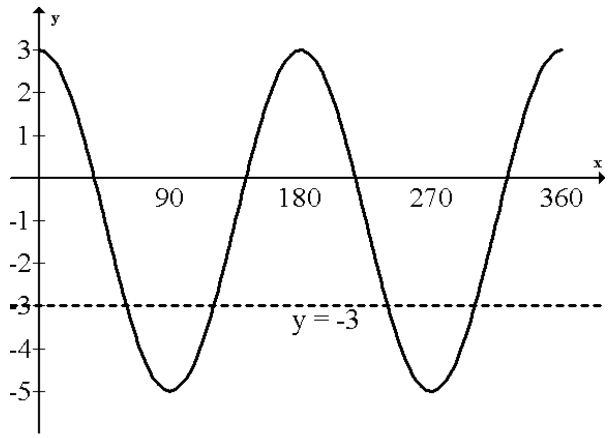
3.(a) The diagram shows the graph of $y = a\sin bx$.
Write down the values of a and b.

(b) Find the coordinates of P and Q the points of intersection of this graph and the line y = 2.



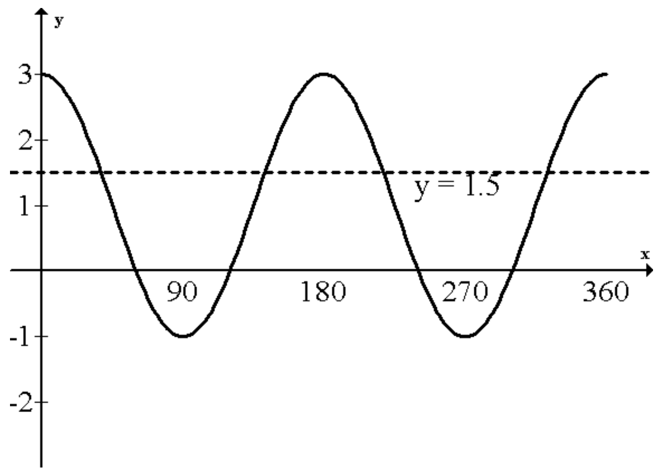
4. (a) The diagram shows the graph of $y = a\cos bx + c$.
Write down the values of a, b and c.

(b) Find the coordinates of the points of intersection of this graph and the line y = -3, $0 \le x \le 360$



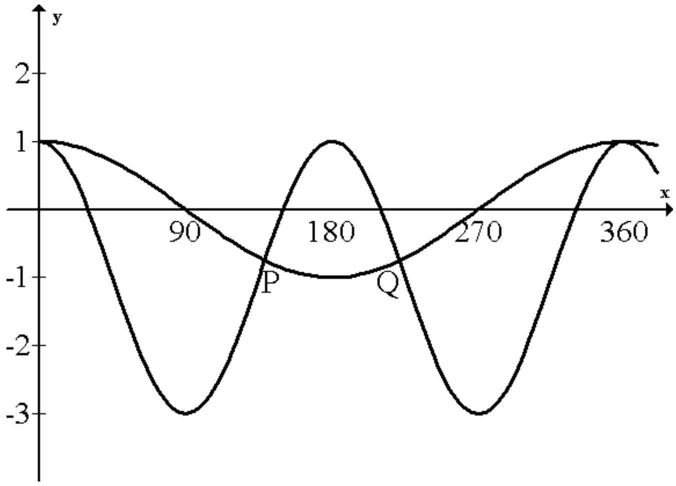
5. (a) The diagram shows the graph of $y = a\cos bx + c$.
Write down the values of a, b and c.

(b) For the interval $0 \le x \le 360$, find the points of intersection of this graph and the line y = 1.5



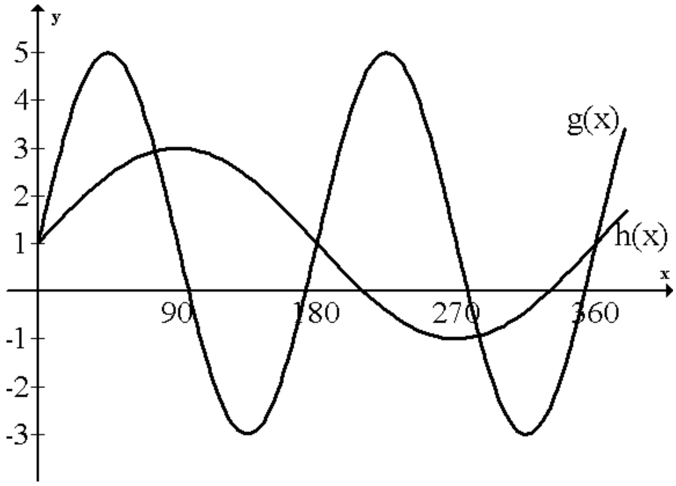
6. The diagram shows the graphs of
   g(x) = acos bx + c and h(x) = cos x

   (a) State the values of a, b and c.
   (b) Find the coordinates of P and Q.



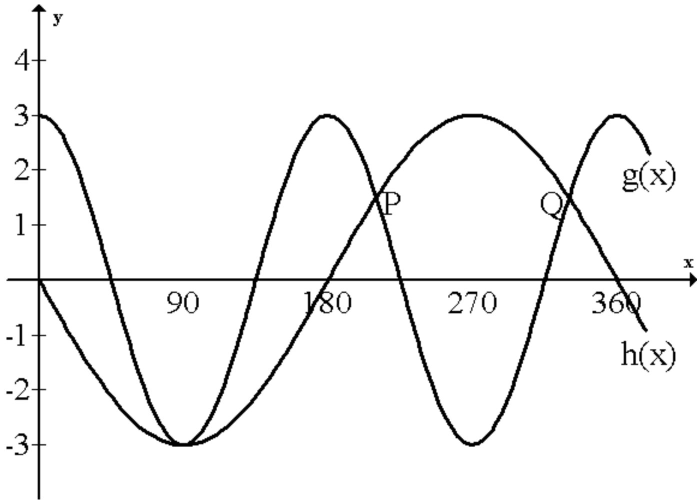
7. The diagram shows the graphs of
   g(x) = asin bx + c and h(x) = dsin x + e

   (a) Write down the values of a, b and c.
   (b) Write down the values of d and e.
   (c) Find the points of intersection of these curves for $0 \le x \le 360$



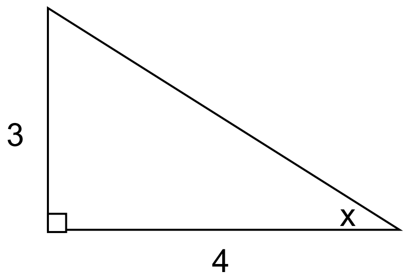
8. The diagram shows the graphs of
   h(x) = asin x and g(x) = bcos cx.

   (a) Write down the values of a ,b and c.
   (b) Find the coordinates of P and Q.

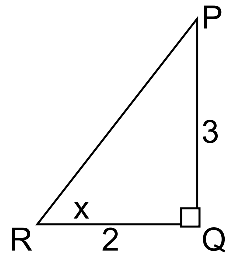


Addition / Double Angle Formulae
Applications

1. Using the triangle shown opposite, show that the exact value of cos 2x is $\frac{7}{25}$



2. Using triangle PQR, find the exact value of sin 2x.



3. Given $\sin x = \frac{2}{\sqrt{5}}$, find the exact value of

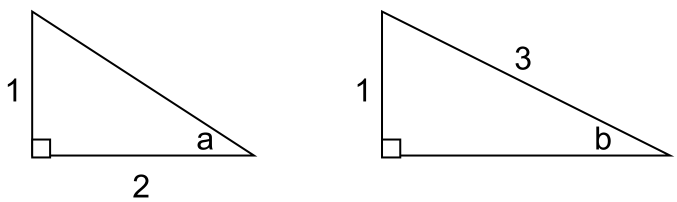
(a) sin 2x
(b) cos 2x
(c) tan 2x

4. Given $\sin x = \frac{1}{3}$

(a) Show that (i) $\cos 2x = \frac{7}{9}$ (ii) $\sin 2x = \frac{4\sqrt{2}}{9}$
(b) By writing sin 4x as sin 2(2x), find the exact value of sin 4x

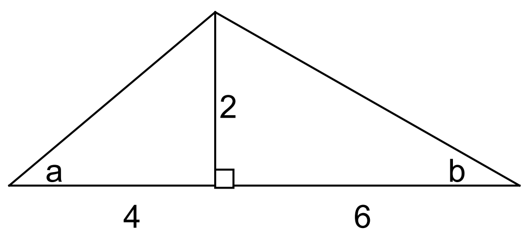
5. Using the triangles opposite show that

$\sin(a - b) = \frac{2\sqrt{2} - 2}{3\sqrt{5}}$



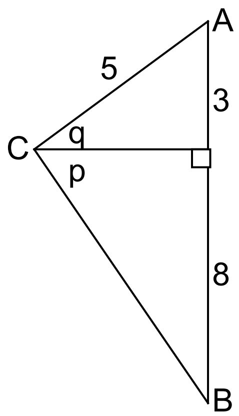
6. Using the diagram shown show that

$\sin(a + b) = \frac{1}{\sqrt{2}}$



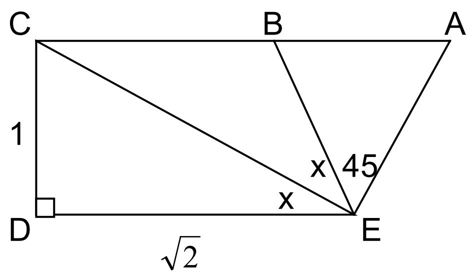
7. The diagram shows triangle ABC.
   Find the exact value of

   (a) sin ACB
   (b) cos ACB
   (c) tan ACB



8. In the diagram angle DEC = angle CEB = x^0
   CD = 1 unit and DE = $\sqrt{2}$ units.

   Find the exact value of cos DEA.



9. Functions $f(x) = \sin x$, $g(x) = x + \frac{\pi}{6}$ and $h(x) = x - \frac{\pi}{6}$

   (a) Show that $f(g(x)) = \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x$

   (b) Find a similar expression for f(h(x)).

   (c) Hence solve the equation $f(g(x)) + f(h(x)) = \frac{3}{2}$ for $0 \le x \le 2\pi$